Enterprise investment under price-sensitive demand

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Abstract. This paper proposed an equity financing investment model and a debt financing investment model based on real options theory, with considering price-sensitive demand and price obeying the Geometric Brownian Motion. We derived the optimal solutions, which show that the optimal bankruptcy threshold of debt financing is higher than equity financing, the optimal bankruptcy threshold and the optimal investment threshold decrease as the market capacity increases, and increase as the price-sensitive coefficient increases.

Introduction

With more uncertainty of market environment, a growing number of scholars study investment based on real options theory. Dixit and Pindyck (1994) proposed that uncertain investment has three characteristics: (1) irreversibility; (2) delayability; (3) uncertainty over the future returns. The returns on invested assets are not only related to price, but also to demand. In general, demand decreases as price rises, and vice versa. So how does this inverse relationship affect investment? Considering price-sensitive demand, this paper built up two investment models. By solving the models, the optimal bankruptcy threshold expression and the optimal investment threshold equation were obtained.

Related literature. As far as I know, McDonald and Siegel (1986) considered the relationship of demand and price to study investment earlier, they constructed a nonlinear function of price about demand, and obtained that the option value increases as the demand uncertainty increases in the example of a monopolist investment decision. Subsequently, Kulatilaka and Perotti (1998) constructed a linear function of price about demand and demand random increasing variable, and proved that demand uncertainty would cause enterprises to postpone the execution of growth options. Recently, some scholars constructed a function of price about demand, which is like to Kulatilaka and Perotti's (1998) to study capacity investment, they demonstrated that enterprises with flexible capacity would implement capacity expansion earlier (Dangl (1999), Hagspiel V, etc. (2016), De and Massabò (2018)). Different from the above literature, this paper constructed a function of demand about price to study uncertain investment.

Model

Considering that an enterprise has a growth option, the enterprise will pay for the investment cost through equity financing or debt financing, setting the investment cost as I. The enterprise executes the growth option according to the market price, and setting the market price as \( p \), assuming the market price obeys the Geometric Brownian Motion, then

\[
dp = \mu dt + \sigma dW
\]  

(1)

Where \( \mu \) is called the drift parameter, and \( \sigma \) the variance parameter, \( dW \) is the increment of the Wiener process: \( dW = \varepsilon(t)(dt)^{1/2} \), \( \varepsilon(t) \) is the continuous random variable on \( t \), \( \varepsilon(t) \sim N(0,1) \) and \( \varepsilon(i), \varepsilon(j) = 0 \) for any \( i, j \).  

Because demand decreases as price rises, assuming demand about price is a linear reduction function: \( D(p) = a - bp \), where \( a \) is the demand when \( b = 0 \), which is called market capacity, \( b \) is the
price-sensitive coefficient of the demand. Setting \( \tau \) as the corporate tax rate and \( Z \) as the operating cost, then the after-tax net income is

\[
R(p) = (1-\tau)[p(a-b)p-Z]
\]

(2)

Setting other symbols: \( r \)=discount rate, in order to ensure the bankruptcy threshold is positive, assuming \( r>2\mu+\sigma^2; p_i \)= the optimal investment threshold under equity financing; \( p_c \)=the optimal bankruptcy threshold under equity financing; \( p_a \)= the optimal investment threshold under debt financing; \( p_d \)=the optimal bankruptcy threshold under debt financing; \( c \)= the debt payment for each period; \( T_i \)= the time when the enterprise executes the growth option; \( T_s \)= the time when the enterprise executes the bankruptcy.

**Equity financing.**

Considering that the investment cost comes from equity financing when the enterprise executes the growth option, then the expected value of the enterprise before bankruptcy is

\[
V(p) = E_p \left[ (1-\tau) \left[ p(a-b)p-Z \right] e^{-\tau(t-s)} \right] p \geq p_s
\]

(3)

Equation (3) is a dynamic programming, which represents the total net income in \([t, T_s]\). Equation (3) can be equivalently converted to the ordinary differential equation (Pindyck (1990)):

\[
rV(p) = (1-\tau)[p(a-b)p-Z]+\mu p'V(p)+\sigma^2V''(p)/2 \quad p \geq p_s
\]

(4)

The investment cost comes from equity financing, so bankruptcy is due to operating deficit. After bankruptcy liquidation, the enterprise value is zero (Brennan, Schwartz (1984)), that is: \( V(p_s) = 0 \). Therefore, the optimal bankruptcy threshold satisfies the following conditions:

\[
V(p_s) = 0
\]

(5)

\[
V'(p_s) = 0
\]

(6)

Where, the equation (6) is the smooth-pasting condition.

**Proposition 1.** If financing is through equity, the enterprise value before bankruptcy is

\[
V(p) = H(p) - H(p_s)(p/p_s)^\beta \quad p \geq p_s
\]

(7)

Where,

\[
H(p) = (1-\tau) \left[ \frac{ap}{r-\mu} - \frac{bp^2}{r-2\mu-\sigma^2} - \frac{Z}{r} \right]
\]

(8)

\[
\beta = \frac{1}{\sigma} \left[ \left( \frac{\mu-\sigma^2}{\sigma} \right) - \sqrt{\left( \frac{\mu-\sigma^2}{\sigma} \right)^2 + 2\sigma^2} \right] > 0
\]

(9)

\[
p_s = (-B+\sqrt{B^2-4AC})/(2A)
\]

(10)

\[
A = b\beta^2 -(r-\mu), B = a\beta -(r-\mu)(\sigma^2+2\mu)r, C = 2\beta \sigma^2 / (r-2\mu-\sigma^2)
\]

Proof. Because the general solution of the equation (4) is: \( V(p) = H(p) - C (p/p_s)^\beta \), according to \( V(p_s) = 0, C = H(p_s) \) can be obtained, and since \( V'(p) = H(p) - H(p_s)(\beta/p_p)' \), so \( V(p_s) = H(p_s) - H(p_s)(\beta/p_s) \). Because \( V(p_s) = 0, p_s = [-B+(B^2-4AC)^{1/2}]/(2A), [-B-(B^2-4AC)^{1/2}]/(2A) \). At last, since \( A < 0 \), so \( p_s = [-B+(B^2-4AC)^{1/2}]/(2A) \).

Considering that the enterprise has a growth option, the equity value before executing the growth option (0<T_i<p<T_p) satisfies the following model:

\[
rE_i(p) = \mu pE_i(p) + p'\sigma^2 E_i(p)/2 \quad p \leq p_i
\]

(11)

\[
E_i(p) = V(p) - 1
\]

(12)

**Proposition 2.** The equity value before executing the growth option is

\[
E_i(p) = (V(p) - 1)(p/p)\gamma \quad p \leq p_i
\]

(13)

Where,

\[
\gamma = \frac{1}{\sigma} \left[ \left( \frac{\mu-\sigma^2}{\sigma} \right) + \sqrt{\left( \frac{\mu-\sigma^2}{\sigma} \right)^2 + 2\sigma^2} \right] > 1
\]

\[
p_i \text{ satisfies the following equation:}
\]

\[
(H(p)-1)\gamma - (1-\tau) \left[ \frac{ap}{r-\mu} - \frac{2bp^2}{r-2\mu-\sigma^2} \right] = (\gamma - \beta) H(p_s) \left( \frac{p_i}{p_s} \right)^\beta
\]

(14)
Therefore, the optimal bankruptcy threshold satisfies the following conditions:

\[
\beta = \frac{\gamma}{\pi} > 1 \quad \text{when} \quad \frac{\pi}{\rho} \cdot \frac{\rho}{\rho^*} = \frac{\pi}{\rho^*} > 1
\]

Since \( C = \frac{\beta}{\rho} \), the debt value is equal to the after-tax discount value of assets, that is, equation (14) must have a real solution.

The expected value of the equity before executing the bankruptcy is

\[
E(p) = E[p \cdot (p-bp) - Z - c]e^{-(r+\sigma^2)ds} \quad p \geq \rho
\]

Equation (15) can be equivalently converted to the ordinary differential equation:

\[
rE(p) = -\beta + \pi p + \beta_0 E(p) + p^2\sigma^2 E''(p)/2 \quad p \geq \rho
\]

After the enterprise performs bankruptcy liquidation, the equity value is zero, that is: \( E(p_{cs}) = 0 \).

Therefore, the optimal bankruptcy threshold satisfies the following conditions:

\[
E(p_{cs}) = 0
\]

Equation (17)

\[
E(p_{cs}) = 0
\]

Equation (18)

**Proposition 3.** If financing is through debt, then the equity value before bankruptcy is

\[
E(p) = H(p) - \frac{1}{r} \left[ H(p_{cs}) - \frac{1}{r} \left( \frac{p}{\rho_{cs}} \right)^{\gamma} \right] \quad p \geq \rho
\]

Where,

\[
p_{cs} = \left( -B + \sqrt{B^2 - 4AC_c} \right)/(2A) > \rho
\]

\[
C_c = \beta(Z + c)(r-\mu)(r-2\mu - \sigma^2).
\]

The proof of the equation (19) is similar to the **Proposition 1**, so it is omitted. For equation (20), since \( C_c < 0 \Rightarrow (B^2 - 4AC_c)^{1/2} < (B^2 - 4AC_c)^{1/2} \Rightarrow [-B + (B^2 - 4AC_c)]^{1/2}/(2A) > [-B + (B^2 - 4AC_c)]^{1/2}/(2A) \), so \( p_{cs} > p_s \).

The expected of the debt value before bankruptcy is

\[
D(p) = E[p \cdot (p-bp) - Z-c]e^{-(r+\sigma^2)ds} \quad p \geq \rho
\]

Equation (21)

\[
rD(p) = c + \mu p D(p) + p^2\sigma^2 D''(p)/2 \quad p \geq \rho
\]

The investment cost comes from debt financing, and bankruptcy is due to insolvency. From the perspective of creditors, the debt value is equal to the after-tax discount value of assets, that is,

\[
D(p_{cs}) = H(p_{cs})\]

\[
\lim_{p \to \infty} D(p) = c/r
\]

Where, equation (24) represents the debt value is equal to the perpetual value of debt payment, when \( p \to \infty \).

**Proposition 4.** The debt value before bankruptcy is

\[
D(p) = \frac{c}{r} \left[ \frac{c}{r} - H(p_{cs}) \right] \left( \frac{p}{\rho_{cs}} \right)^{\gamma} \quad p \geq \rho
\]

The proof. Because the general solution of equation (25) is: \( D(p) = c/r - C(p/ho_{cs})^\gamma \), according to \( D(p_{cs}) = H(p_{cs}) \), \( C = c/r - H(p_{cs}) \) can be obtained. Since \( \lim_{p \to \infty} D(p) = c/r = \beta = 1/\sigma^2 \{-\mu/\sigma^2 - 2\sigma^2/2\}^{1/2} < 0 \), so the **Proposition 4** is established.

Because enterprise value is the sum of equity value and debt value, according to equations (19) and (25), the enterprise value is
The equity value before executing the growth option satisfies the following model:

\[ rE_\alpha(p) = \mu pE_\alpha(p) + \frac{1}{2} p^2 \sigma^2 E_\alpha(p) \quad p \leq p_\alpha \]

(27)

**Proposition 5.** The equity value before executing the growth option is

\[ E_\alpha(p) = (V(p_\alpha) - 1) \left( \frac{p}{p_\alpha} \right) \quad p \leq p_\alpha \]

(28)

Where \( p_t \) satisfies the following equation:

\[ \left( H(p_\alpha) + \frac{\tau c}{r} \right) \left( 1 - \frac{r p}{r - \mu} \right) = \left( r - \beta \right) \left( \frac{p}{p_\alpha} \right)^\beta \]

(29)

According to the proof of **Proposition 2**, we can easily get the conclusion of **Proposition 5**.

**Sensitivity analysis**

**Proposition 6.** Whether it is equity financing or debt financing, the optimal bankruptcy threshold decreases with the increase of market capacity, and increases with the increase of price-sensitive coefficient.

**Proof.** Let \( M_1 = -2r(\beta - 2)(r-\mu) \), \( M_2 = 4rZf(\beta - 2)(r-\mu)^2 \), \( M_3 = r(\beta - 1)(\sigma^2 - 2\mu - r) \), \( M_4 = 4AC/M_3^2 \) and \( M_5 = -M_3/(2A) \). From equation (10), it can be got that

\[ p_0 = \frac{B - \sqrt{B^2 - M_5}}{M_5} \Rightarrow dp_0 \over db = \frac{2B^2 - 2B \sqrt{B^2 - M_5} - M_5}{2M_5 \sqrt{B^2 - M_5}} > 0 \]

(30)

Similarly, let \( M_6 = 4r(Z+c)(\beta - 2)(r-\mu)^2 \) and \( M_7 = 4AC_c / M_3^2 \). From equation (20), it can be got that

\[ dp_0 \over da = \frac{B - \sqrt{B^2 - M_7}}{2M_7 \sqrt{B^2 - M_7}} > 0, \quad dp_0 \over da = M_4 \left( 1 - \frac{a}{\sqrt{a^2 - M_4}} \right) < 0 \]

For the analysis of the optimal investment threshold, we will use numerical examples to analyze.

(1) Let \( r = 6\%, \mu = 1, I = 50, \tau = 20\%, Z = 1, \sigma = 0.2, c = 5 \) and \( a = 10 \).

According to equations (14) and (29), the relationship between \( p_0, p_{ci} \) and \( b \) is showed in **Fig.1**. It can be seen from **Fig.1** that the optimal investment threshold increases with the increase of price-sensitive coefficient, under the same price-sensitive coefficient, the optimal investment threshold of equity financing is higher than debt financing.

(2) Let \( r = 6\%, \mu = 1, I = 50, \tau = 20\%, Z = 1, \sigma = 0.2, c = 5 \) and \( b = 1 \).
According to equations (14) and (29), the relationship between $p_t$, $c_t$, and $a$ is showed in Fig.2. It can be seen from Fig.2 that the optimal investment threshold decreases with the increase of market capacity; under the same market capacity, the optimal investment threshold of equity financing is higher than debt financing.

Conclusion

Our research shows that price-sensitive demand and financing methods have an impact on investment: The optimal bankruptcy threshold of debt financing is higher than equity financing, the optimal bankruptcy threshold and the optimal investment threshold decrease as the market capacity increases, and increase as the price-sensitive coefficient increases. Consequently seen: The bankruptcy risk of debt financing is higher than the equity financing, the bankruptcy risk of the enterprise with higher market capacity is lower, and the enterprise with a higher price-sensitive coefficient has a higher risk of bankruptcy.

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