Existence of solutions to boundary value problem of fourth-order with functional boundary conditions at resonance

Fei Yang¹,a, Yuanjian Lin²,b

¹Nanchang Institute of Science and Technology, Nanchang 330108, Jiangxi
²Nanchang Institute of Science and Technology, Nanchang 330108, Jiangxi

afeixu126@126.com, blinyuanzhou@126.com

Keywords: Functional boundary condition; fourth-order; resonance; Boundary value problem.

Abstract: We study the existence of solutions for a fourth-order functional boundary value problem at resonance

\[ u^{(4)}(t) = f(t, u(t), u'(t), u''(t), u'''(t)), t \in (0,1) \]

\[ \varphi_i(u) = \varphi_i(u) = \varphi_i(u) = \varphi_i(u) = 0 \]

where \( \varphi_i : C^3[0,1] \to R, i = 1, 2, 3 \). By using the coincidence degree theory due to Mawhin and constructing suitable operators.

1. Introduction and introduction

A boundary value problem is said to be at resonance if the corresponding homogeneous boundary value problem has a non-trivial solution. Boundary value problems at resonance have been studied by many authors. We refer the readers to [1-9] and the references cited therein. In [10], the authors discussed the second-order differential equation

\[ x''(t) = f(t, x(t), x'(t)), t \in (0,1) \]

with functional boundary conditions \( \Gamma_1(x) = 0, \Gamma_2(x) = 0 \), where \( \Gamma_1, \Gamma_2 \) are linear functional on \( C^1[0,1] \) satisfying the general resonance condition \( \Gamma_1(x) = \Gamma_2(x) \).

In [11] proved the existence of solutions for third-order functional boundary value problems (FBVPs) at resonance

\[ x'''(t) = f(t, x(t), x'(t), x''(t)), 0 < t < 1 \]

\[ \varphi_1(x) = \varphi_2(x) = \varphi_3(x) = 0, \]

where \( \varphi_i : C^3[0,1] \to R, i = 1, 2, 3, 4 \) are bounded linear functionals. In this paper, the existence of solutions to the following boundary value problems is studied by using the coincidence degree extension theorem

\[ u^{(4)}(t) = f(t, u(t), u'(t), u''(t), u'''(t)), t \in (0,1) \]

\[ \varphi_i(u) = \varphi_i(u) = \varphi_i(u) = \varphi_i(u) = 0 \]

(1)

where \( \varphi_i : C^3[0,1] \to R, i = 1, 2, 3, \varphi_i(t^j) = 0, i = 1, 2, 3, 4, j \in \{1, 2, 3, 4\} \).

2. Preliminaries

For convenience, we denote

\[\Delta = \begin{bmatrix} \varphi_1(t^3) & \varphi_1(t^2) & \varphi_i(t) & \varphi_i(l) \\ \varphi_2(t^3) & \varphi_2(t^2) & \varphi_i(t) & \varphi_i(l) \\ \varphi_3(t^3) & \varphi_3(t^2) & \varphi_i(t) & \varphi_i(l) \\ \varphi_4(t^3) & \varphi_4(t^2) & \varphi_i(t) & \varphi_i(l) \end{bmatrix} \]
From the last three determinants we can define and derive the following three relations:

\[
\Delta_1(v) = \begin{vmatrix}
\varphi_1(t^3) & \varphi_1(t^2) & \varphi_1(t) & \varphi_1(1) \\
\varphi_2(t^3) & \varphi_2(t^2) & \varphi_2(t) & \varphi_2(1) \\
\varphi_3(t^3) & \varphi_3(t^2) & \varphi_3(t) & \varphi_3(1) \\
\varphi_4(t^3) & \varphi_4(t^2) & \varphi_4(t) & \varphi_4(1) \\
\end{vmatrix},
\]

\[
\Delta_2(v) = \begin{vmatrix}
\varphi_1(t^3) & \varphi_1(t^2) & \varphi_1(t) & \varphi_1(1) \\
\varphi_2(t^3) & \varphi_2(t^2) & \varphi_2(t) & \varphi_2(1) \\
\varphi_3(t^3) & \varphi_3(t^2) & \varphi_3(t) & \varphi_3(1) \\
\varphi_4(t^3) & \varphi_4(t^2) & \varphi_4(t) & \varphi_4(1) \\
\end{vmatrix},
\]

\[
\Delta_3(v) = \begin{vmatrix}
\varphi_1(t^3) & \varphi_1(t^2) & \varphi_1(t) & \varphi_1(1) \\
\varphi_2(t^3) & \varphi_2(t^2) & \varphi_2(t) & \varphi_2(1) \\
\varphi_3(t^3) & \varphi_3(t^2) & \varphi_3(t) & \varphi_3(1) \\
\varphi_4(t^3) & \varphi_4(t^2) & \varphi_4(t) & \varphi_4(1) \\
\end{vmatrix},
\]

\[
\Delta_4(v) = \begin{vmatrix}
\varphi_1(t^3) & \varphi_1(t^2) & \varphi_1(t) & \varphi_1(1) \\
\varphi_2(t^3) & \varphi_2(t^2) & \varphi_2(t) & \varphi_2(1) \\
\varphi_3(t^3) & \varphi_3(t^2) & \varphi_3(t) & \varphi_3(1) \\
\varphi_4(t^3) & \varphi_4(t^2) & \varphi_4(t) & \varphi_4(1) \\
\end{vmatrix},
\]

From the last three determinants we can define and derive the following three relations:

\[
\Delta_1(Lu) = \begin{vmatrix}
\varphi_1\left(-u''(0)t^3 - 2u''(0)t^2 - 2u'(0)t - 2u(0)\right) & \varphi_1(t^3) & \varphi_1(t^2) & \varphi_1(t) & \varphi_1(1) \\
\varphi_2\left(-u''(0)t^3 - 2u''(0)t^2 - 2u'(0)t - 2u(0)\right) & \varphi_2(t^3) & \varphi_2(t^2) & \varphi_2(t) & \varphi_2(1) \\
\varphi_3\left(-u''(0)t^3 - 2u''(0)t^2 - 2u'(0)t - 2u(0)\right) & \varphi_3(t^3) & \varphi_3(t^2) & \varphi_3(t) & \varphi_3(1) \\
\varphi_4\left(-u''(0)t^3 - 2u''(0)t^2 - 2u'(0)t - 2u(0)\right) & \varphi_4(t^3) & \varphi_4(t^2) & \varphi_4(t) & \varphi_4(1) \\
\end{vmatrix} = -u''(0)\Delta \quad (2)
\]

\[
\Delta_2(Lu) = \begin{vmatrix}
\varphi_1\left(-u''(0)t^3 - 2u''(0)t^2 - 2u'(0)t - 2u(0)\right) & \varphi_1(t^3) & \varphi_1(t^2) & \varphi_1(t) & \varphi_1(1) \\
\varphi_2\left(-u''(0)t^3 - 2u''(0)t^2 - 2u'(0)t - 2u(0)\right) & \varphi_2(t^3) & \varphi_2(t^2) & \varphi_2(t) & \varphi_2(1) \\
\varphi_3\left(-u''(0)t^3 - 2u''(0)t^2 - 2u'(0)t - 2u(0)\right) & \varphi_3(t^3) & \varphi_3(t^2) & \varphi_3(t) & \varphi_3(1) \\
\varphi_4\left(-u''(0)t^3 - 2u''(0)t^2 - 2u'(0)t - 2u(0)\right) & \varphi_4(t^3) & \varphi_4(t^2) & \varphi_4(t) & \varphi_4(1) \\
\end{vmatrix} = -3u''(0)\Delta \quad (3)
\]

\[
\Delta_3(Lu) = \begin{vmatrix}
\varphi_1\left(-u''(0)t^3 - 2u''(0)t^2 - 2u'(0)t - 2u(0)\right) & \varphi_1(t^3) & \varphi_1(t^2) & \varphi_1(t) & \varphi_1(1) \\
\varphi_2\left(-u''(0)t^3 - 2u''(0)t^2 - 2u'(0)t - 2u(0)\right) & \varphi_2(t^3) & \varphi_2(t^2) & \varphi_2(t) & \varphi_2(1) \\
\varphi_3\left(-u''(0)t^3 - 2u''(0)t^2 - 2u'(0)t - 2u(0)\right) & \varphi_3(t^3) & \varphi_3(t^2) & \varphi_3(t) & \varphi_3(1) \\
\varphi_4\left(-u''(0)t^3 - 2u''(0)t^2 - 2u'(0)t - 2u(0)\right) & \varphi_4(t^3) & \varphi_4(t^2) & \varphi_4(t) & \varphi_4(1) \\
\end{vmatrix} = -6u'(0)\Delta \quad (4)
\]
and \( \Delta_4(Lu) = -6u(0)\Delta. \) Also, \( \Delta_y, i, j = 1, 2, 3, 4, \Delta_k(v), i, k = 1, 2, 3, 4, j = \{1, 2, 3, 4\} \setminus \{k\} \), are the cofactors of \( \varphi(t^{i-j}) \) in \( \Delta, \Delta_k(v), k = 1, 2, 3, 4 \) respectively.

Mawhin’s continuation theorem:
Let \( X, Y \) be the Banach space, \( L : domL \subset X \rightarrow Y \) be the Linear mapping, \( N : X \rightarrow Y \) be the Nonlinear continuous mapping, \( \dim ker L = \dim \langle \frac{Y}{\text{Im} L} \rangle < +\infty \), and \( \text{Im} L \) is a Closed set in \( Y \), according to \( L \) is a Fredholm operator whose index is zero. If \( L \) is a Fredholm operator whose index is zero, then there is a continuous projection operator \( P : X \rightarrow \ker L \) and \( \text{Im} L \rightarrow \text{Im} Q \). Let \( L_P = L \big|_{\text{dom} L \cap X} \) is invertible, so let’s call that the inverse \( K \). If \( QN(\Omega) \) is bounded, and \( K(I - Q)N : \Omega \rightarrow X \) is relatively tight in \( X \), according to \( N \) is \( L \)-tight in \( \Omega \), where \( \Omega \) is any bounded open set in \( X \).

**Theorem 2.1:** (Mawhin coincidence degree theory \cite{10}) Let \( X, Y \) be the Banach space, \( L \) is a Fredholm operator whose index is zero, \( N : \Omega \rightarrow Y \) is \( L \)-tight in \( \Omega \). If

\[
(1) \quad Lx \neq \lambda Nx, \forall (x, \lambda) \in (\text{dom} L \cap \partial \Omega) \times (0, 1);
\]

\[
(2) \quad Nx \notin \text{Im} L, \forall x \in \ker L \cap \partial \Omega;
\]

\[
(3) \quad \deg(NQ, \Omega \cap \ker L, 0) \neq 0 \text{, there } J : \text{Im} Q \rightarrow \ker L \text{ is a linear isomorphism; equation } Lx = Nx \text{ has at least one solution in } \text{dom} L \cap \Omega.
\]

In this paper, we always suppose that the following condition holds:

\[\text{(C)} \quad \text{There exist constants } k_i > 0, i = 1, 2, 3, 4, \text{ such that } |\varphi(u)| \leq k_i |u|, u \in U \text{ and the function } f(t, x, y, z, w) \text{ satisfies the Carathéodory conditions, that is, } f(\cdot, x, y, z, w) \text{ is measurable for each fixed } (x, y, z, w) \in \mathbb{R}^3 \text{, } f(t, \cdot, \cdot, \cdot, \cdot) \text{ is continuous for a.e. } t \in [0, 1].\]

3. The main results

In this case, we assume that there exists \( j \in \{1, 2, 3, 4\} \) such that \( \Delta_{j4} \neq 0 \). In what follows, we choose and fix such \( j \).

**Lemma 3.1** \cite{12} There exists a function \( g_4 \in V \) such that \( \Delta_4(g_4) = 1 \).

**Lemma 3.2** \cite{12} \( \text{Im} L = \{v \in V : \Delta_4(v) = 0\} \).

**Lemma 3.3** \( K_{P4} = (L \big|_{\text{dom} L \cap \ker L})^{-1} \).

We introduce the constants \( l_1 = k_1 |\Delta_{14}| + k_2 |\Delta_{24}| + k_3 |\Delta_{34}| + k_4 |\Delta_{44}| \) and

\[ l = \max \{k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8, k_9, k_{12}\}. \quad (5) \]

The next assumption is fulfilled in the main results by virtue of appropriate assumptions on \( f(t, \cdot, \cdot, \cdot, \cdot) \):

\( (H_1) \) For any \( r > 0 \), there exists a function \( h_r \in V \) such that \( |f(t, u(t), u'(t), u''(t))| \leq h_r(t), \quad u \in U, \|u\| \leq r. \)

**Lemma 3.4** There exists a function \( g_4 \in V \) such that \( \Delta_4(g_4) = 1 \).

If \( (H_1) \) holds and \( \Omega \subset U \) is bounded, then \( N \) is \( L \)-compact on \( \Omega \).

In order to obtain the main results, we impose the following conditions:

\( (H_2) \) There exist nonnegative functions \( a, b, c, d, e \in V \) such that
\[ f(t, x, y, z, w) \leq a(t) + b(t) |x| + c(t) |y| + d(t) |z| + e(t) |w|, \quad t \in [0, 1], a, b, c, d, e \in R; \]

(H3) There exists a constant \( M_{04} > 0 \) such that \( \Delta_{44} (Nu) \neq 0 \) if \( u(t) > M_{04}, t \in [0, 1] \);

(H4) There exists a constant \( M_{14} > 0 \) such that if \( |c| > M_{14}, \) then one of the following two inequalities holds:

\[ c\Delta_{44} (Nu) > 0 \quad \text{or} \quad c\Delta_{44} (Nu) < 0 \]

(here \( Nu = f(t, c, 0, 0, 0), c \in R \))

Lemma 3.5 \[12\] Assume that (H2) (H3) hold and let

\[ A_{p4}(\|p\| + \|\ell\| + \|\eta\| + \|\sigma\|) < \frac{1}{2}. \]

where \( A_{p4} = 1 + \frac{8l}{\Delta_{44}}. \) Then \( \Omega_{44} = \{u \in \text{dom} L \setminus \text{Ker} L : Lu = \lambda Nu, \lambda \in (0, 1)\} \) is bounded.

Lemma 3.6 \[12\] Assume that (H4) holds. Then \( \Omega_{24} = \{u \in \text{Ker} L : Nu \in \text{Im} L\} \) is bounded.

Lemma 3.7 Assume that (H4) holds. Then \( \Omega_{44} = \{u : \rho \lambda U + (1 - \lambda)\Delta_{44} (Nu) = 0, u \in \text{Ker} L, \lambda \in [0, 1]\} \) is bounded, where \( \rho = \begin{cases} 1, & \text{if (6) holds} \\ -1, & \text{if (7) holds} \end{cases} \).

Theorem 3.1: Assume that (H2)-(H4) and (8) hold. Then problem (1) has at least one solution.

Proof Let \( \Omega \supset \bar{\Omega}_{14} \cup \bar{\Omega}_{24} \cup \bar{\Omega}_{34} \cup \bar{\Omega}_{44} \) be bounded. It follows Lemmas3.5 and Lemmas3.6 that

\( Lu \neq \lambda Nu, u \in (\text{dom} L \setminus \text{Ker} L) \cap \partial \Omega, \lambda \in (0, 1), \) and \( Nu \in \text{Im} L, u \in \text{Ker} L \cap \partial \Omega. \)

Let

\( H(u, \lambda) = \lambda \rho u + (1 - \lambda)J_{44} Nu, \)

where \( J_{44} : \text{Im} Q_{44} \rightarrow \text{Ker} L \) is an isomorphism defined by \( J_{44}(cg_{4}) = c, c \in R. \) By Lemma3.7, we know

\( H(u, \lambda) \neq 0, u \in \partial \Omega \cap \text{Ker} L, \lambda \in [0, 1]. \) Since the degree is invariant under a homotopy,

\[ \deg(J_{44} Nu, \Omega \cap \text{Ker} L, 0, 0) = \deg(H(\cdot, 0), \Omega \cap \text{Ker} L, 0, 0) = \deg(H(\cdot, 1), \Omega \cap \text{Ker} L, 0, 0) = \deg(\rho I, \Omega \cap \text{Ker} L, 0, 0) \neq 0. \]

By Theorem 2.1, \( Lu = Nu \) has a solution in \( \text{dom} L \cap \bar{\Omega}. \)

4. Conclusion

In this paper, the existence of at least one solutions to boundary value problem of resonance fourth-order with functional boundary; By means of Machin’s continuation theorem, the existence of solution is verified.

Acknowledgements

This work is supported by Department of education science and technology research youth project in 2017(Item no.GJJ171107).

References


[2] Chang, SK, Pei, M: Solvability for some higher order multi-point boundary value problems at


