Research on Water Level Management Optimization of the Great Lakes Based on Hierarchical Analysis and Partial Differential Equation Models

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Abstract: This study addresses the water level management challenges in the Great Lakes region by proposing a weight indicator system based on the Analytic Hierarchy Process (AHP) and incorporating Partial Differential Equations (PDEs) to simulate and control the lake water flow and water level variations. By applying the 3δ principle to the water level data spanning the past two decades, the study effectively eliminates outliers, ensuring the accuracy of the data for analysis. The weight indicator system constructed in this research takes into account the needs of various stakeholders, including construction, navigation, ecology, power generation, and agriculture, and develops specific weight matrices for the unique conditions of each lake. The experimental results demonstrate that the model can effectively meet the needs of stakeholders and offers a practical solution for water level control in the Great Lakes. By comparing the actual water level data from 2017 with the model predictions, the model's Mean Squared Error (MSE), Root Mean Squared Error (RMSE), and Mean Absolute Error (MAE) indicators all outperformed the actual data, confirming the model's effectiveness in controlling water levels. These findings underscore the model's potential for practical application and provide a solid foundation for future improvements and refinements. Further research could explore additional factors affecting water levels, such as climate change, extreme weather events, and human activities, to enhance the model's adaptability and accuracy.

1. Introduction

Given the significance of the Great Lakes, the management of their water levels is of paramount importance. Over the years, the Governments of the United States and Canada have collaboratively developed policies and regulations aimed at maintaining the ecological balance and economic utility of the region[1, 2]. However, the task is not without its challenges. Urban runoff, agricultural practices, and industrial waste pose significant threats to the water quality, while the fluctuating water levels themselves present a complex management issue that must consider the needs of various stakeholders, including those in construction, shipping, ecology, power generation, and agriculture[3].

In recent years, the Great Lakes have experienced a range of water level extremes, from record highs to historically low levels[4], which have had profound impacts on the ecosystem and the communities that rely on these waters. These challenges underscore the need for a robust and comprehensive model that can accurately predict and manage water levels to optimize benefits for all stakeholders while minimizing potential adverse effects[5].

In response to this need, we develop a novel water level management model for the Great Lakes. Our approach is grounded in a deep understanding of the interplay between ecological, economic, and social factors, and we leverage the latest advancements in mathematical modeling and statistical analysis to craft a tool that can navigate the intricacies of the Great Lakes system[6].

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2. Water level weighting indicator system

2.1 Competency-based approach

In the 1970s, Saaty, an American operations researcher, proposed a principal official assignment evaluation method, namely, hierarchical analysis. The method decomposes the elements related to decision-making into multiple levels such as objectives, guidelines and programs, and carries out qualitative and quantitative analysis on this basis. And because it is a comprehensive evaluation algorithm with multiple indicators, it is simple and practical, so it is widely used in real life. Based on the method proposed by Saaty, we construct the indicator weight model based on hierarchical analysis to get the optimal water level.

2.2 Data processing

The 3σ principle is based on repeated measurements with equal precision of normal distribution while the interference or noise that causes singular data is difficult to satisfy its distribution. Therefore, we utilize the 3Σ principle to eliminate outliers. The normal distribution function is as follows.

$$f(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}. \tag{1}$$

Its sample data obeys the normal distribution as shown in Fig. 4,it can be assumed that the values of y are almost concentrated in (μ − 3σ , μ + 3σ)

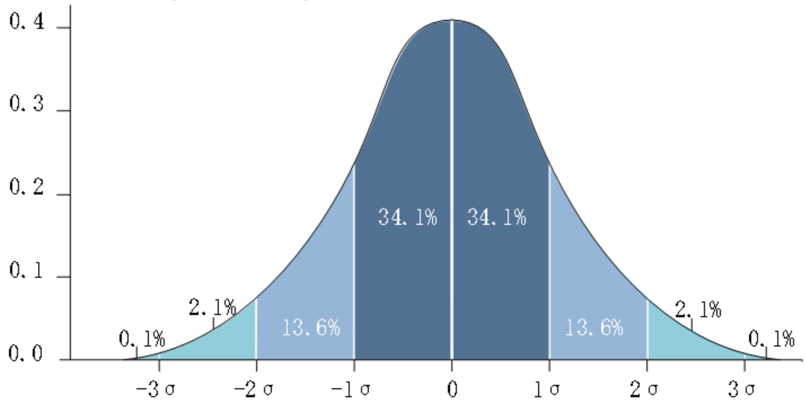

Fig. 1 Normal distribution chart

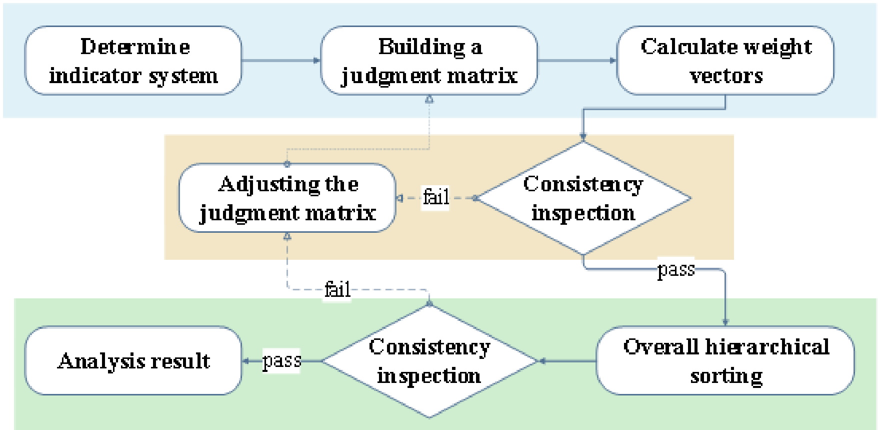

Fig. 2 Multilevel structure flowchart

2.3 Classification of indicators

Hierarchical analysis is to analyze a phenomenon or problem before the phenomenon or problem is decomposed into relevant factors according to their nature, and according to the correlation between them is classified and formed a multilevel structural model, the specific process is shown in Fig. 2 below. In this model, we classify the indicators affecting the water level into five categories, which are building, shipping, ecology, electricity generation, and agriculture.

2.4 Weighting indicator matrix construction

According to the five indicators summarized above, when the lake water level is high, flooding is likely to occur, and when the water level is low, shipping and ecology, etc., are easily affected, so the construction indicator is negatively correlated to the lake water level expectation, and other factors are positively correlated. In order to better balance the interests of all parties, according to the peripheral data of the five lakes consulted, and the importance of the lakeside to the construction, agriculture, ecology, etc., construct different weighting index matrix for different lakes. The specific steps are as follows:

Step1: The lake water matrix construction equation is as follows,for criterion layer A, we can construct a

$$A = \left(a_{ij}\right)_{m\times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}. \tag{2}$$

Where the elements in A satisfy:

$$\begin{cases} a_{ij} > 0 \\ a_{ij} = \frac{1}{aji} \\ a_{ii} = 1 \end{cases}. \tag{3}$$

Step2:Constructing the weight matrix: We based on the evaluation criteria in Fig.1, synthesize the degree of importance of construction, agriculture, etc. around the Great Lakes, construct a matrix to build a weight matrix for the different lakes, and the matrices for each lake are shown in Table 1 below for each lake matrix.

Table 1. The matrices for each lake

	Lake Superior	Lake St.Chair	Lake Erie	Lake Ontario	Lake Michigan
Jan	183.7540929	175.3723962	174.5178617	75.01333357	176.612991
Feb	183.6910494	175.3354397	174.5161226	75.07333357	176.5982083
Mar	183.6567016	175.4397875	174.60134	75.11681183	176.6134257
Apr	183.6949624	175.550657	174.7226443	75.29681183	176.6942953
May	183.7953972	175.6332657	174.8061226	75.46159443	176.7973388
Jun	183.8836581	175.6984831	174.8478617	75.50637704	176.8747301
Jul	183.9436581	175.7237005	174.8330791	75.45420313	176.9064692
Aug	183.9571363	175.6780483	174.7582965	75.31942052	176.8873388
Sep	183.9501798	175.6054397	174.6591661	75.12855096	176.8295127
Oct	183.9336581	175.4963092	174.5648183	74.97768139	176.7612518
Nov	183.8927885	175.4132657	174.5000357	74.91681183	176.7095127
Dec	183.8336581	175.410657	174.4978617	74.92202922	176.6647301

3. Optional water level model

3.1 Situation analysis of the Great Lakes

The direction of water flow between the Great Lakes and the composition of the inflow to each lake can be learned by reviewing the information, in which Lake Superior is replenished by melting snow, precipitation, and water vapor, and all other lakes are replenished by the upstream lakes, in which the flow rate of water from Lake Superior and Lake Ontario can be controlled by two dams. To obtain a better visualization, the relationship between the lakes is graphically represented as shown in Fig. 3 and Fig. 4.

Considering the influence of various factors at the same time, the water depth of each lake is taken as the independent variable and differential equations are introduced to solve the problem.

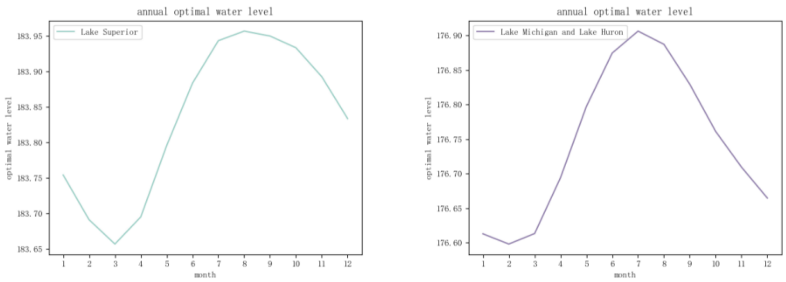

Fig. 3 Lake Superior and Lake Michigan and Huron

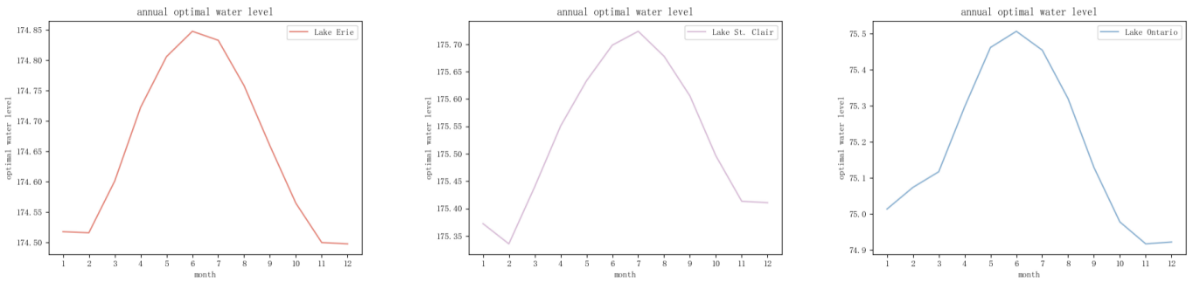

Fig. 4 Lake Erie, Lake St.Clair and Lake Ontario

To facilitate the calculation, the volume of each lake is approximated by the conic equation.

3.2 Partial differential equations

3.2.1 Concepts of partial differential equations

Equations are categorized into algebraic equations, functional equations, and differential equations, where differential equations are further categorized into ordinary differential equations and partial differential equations.

Algebraic equations are equations that contain unknowns, where the unknowns can be one or more. The solution of an algebraic equation is the value of the unknown that makes the equation hold. Algebraic equations can be either univariate equations (containing only one unknown) or multivariate equations (containing multiple unknowns). A quadratic equation is an equation that contains only one unknown, usually in the form $ax + \mathrm{b} = 0$, where a and b are known constants. Methods for solving quadratic equations include shifting terms, combining like terms, and removing coefficients. Multiple equations are equations with multiple unknowns, usually of the form $f(x) = (x_1, x_2, \dots, x_n)$,, where f is a polynomial function. Methods for solving multivariate equations include substitution, elimination, Gaussian elimination, and matrix methods. The solutions of algebraic equations can be in different forms such as real solutions, complex solutions, no solutions, etc. The process of solving algebraic equations requires the use of algebraic operations and equation solving.

A functional equation is an equation or inequality about a function that contains the expression and variables of the function. Functional equations describe the properties and relationships of a function and can be used to find solutions to a function or to study the properties of a function. Functional equations can be categorized into various types, commonly including linear, quadratic, exponential, logarithmic, and trigonometric equations. Each type of functional equation has its own specific solutions and properties. As an example, a linear equation is a functional equation of the form $ax + \mathrm{b} = 0$., where a and b are known constants and x is unknown. The linear equation is solved for the unknown x by shifting terms and simplifying. Another example is a quadratic equation, of the form $ax^2 + bx + c = 0$. Quadratic equations can be solved using, for example, root formulas or collocation methods. Functional equations have a wide range of applications in mathematics and can be used to model, solve real-world problems, and investigate properties of functions.

Partial differential equations are differential equations in which the unknown function is a multivariate function.The partial differential equations alone cannot be solved deterministically, and if they are to constitute a deterministic problem, initial and boundary conditions are also required.

In this case, the boundary conditions specify the information about the values, partial derivatives, etc. of the unknown quantities on the boundary of the partial differential equation. The initial condition specifies information such as the value of the unknown quantity at a particular value taken by an independent variable, the value of the partial derivative, etc.

3.2.2 Partial differential equation (PDE) value solution

The analytic solution of differential equations is generally difficult to find directly; approximation methods must be used to obtain an approximate solution that satisfies a certain degree of accuracy. There are two ways to find the approximate solution of differential equations, one is the approximate analytical method, that is, the use of ordinary differential equations have been familiar with the graded solution method and Picard step-by-step approximation method to give an approximate expression of the solution, known as the approximate analytical method; the other is the numerical solution method, that is, the independent variable in a defined range of a series of discrete points of a set of approximations known as the numerical solution of differential equations in the range of the numerical solution of the process of finding the process of finding numerical solutions of differential equations is called the numerical solution of differential equations. According to the actual situation, this paper will use the numerical integration method to solve the numerical solution of differential equations, and the following focuses on the specific steps of the numerical integration method: numerical integration

$$\begin{cases} \frac{dy}{dx} = f(x, y) \\ y(x_0) = y_0 \end{cases}. \tag{4}$$

Expressed in equivalent integral form:

$$y(x) = y(x_0) + \int_{x}^{x_0} f\,(x, y)dx. \tag{5}$$

$$y(x_1) = y(x_1) + \int_{x_o}^{x_1} f\,(x, y)dx. \tag{6}$$

Using the left rectangular formulation of numerical integration as an approximation to the right end integral,

$$y(x_1) = y(x_0) + f(x_0, y_0)(x_1, x_o) = y(x_0) + hf(x_0, y_0). \tag{7}$$

Thus, the recurrence formula for Euler's method is also obtained

$$y_{n+1} = y_n + hf(x_n, y_n), n = 0,1, \dots, N-1. \tag{8}$$

3.3 Formulation

Applying partial differential equations to this problem, the algorithmic process is shown in Fig. 5.

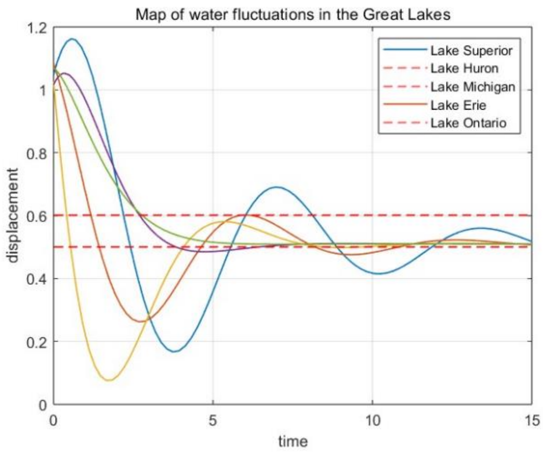

Fig. 5 Water fluctuation steady state diagram

4. Sensitivity Analysis

4.1 Testing sensitivity

In order to test the sensitivity of the control algorithm to the flow rate of the two control dams, the flow rate of the two dams was allowed to increase by 100km/s, or slow down by 100km/s. The results show that the model has a good sensitivity, and the other factors do not have much influence on the model, which means that the model is successful. Meanwhile, Fig. 6 shows that the water level is increasing as the flow rate decreases, which is because the slower the flow rate is, the easier it is for the water to be deposited, resulting in an increase in the water level. According to the graph, when the flow rate of Lake Superior increases by 100km/s Y=0.6129, while when its flow rate decreases by 100km/s, Y=0.6784, which is a small change of 0.0655 before and after; when the flow rate of Lake Ontario increases by 100km/s, Y=0.6164, while when its flow rate decreases by 100km/s, Y=0.6869, which is a small change. The change of 0.0705 before and after is also small. It can be seen that the sensitivity of the model is as expected and can also better satisfy the desires of the stakeholders.

4.2 Test the effectiveness of water level control

In order to test the effectiveness of our established model in controlling the water level, three indicators were chosen to reflect the strengths and weaknesses of the model: the MSE, the RMSE, and the MAE.

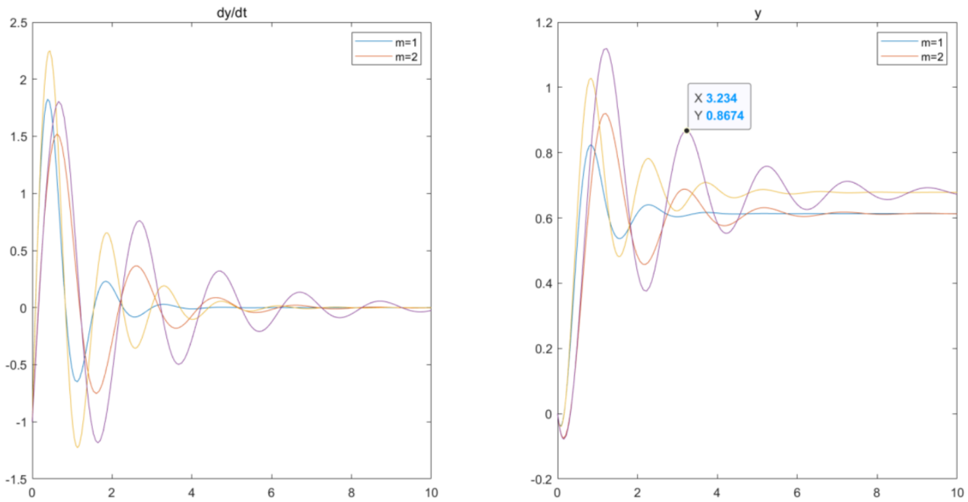

Fig. 6 Model sensitivity oscillations

The smaller the MSE (Mean Squared Error), the better the quality of the model and the more accurate the prediction.

$$MSE = \frac{1}{n}\sum_{i=1}^{n}|\hat{y} - y_i|^2. \quad (9)$$

RMSE (Root Mean Squared Error) reflects the degree of deviation between the real value and the predicted value. the smaller the RMSE, the better the quality of the model and the moreaccurate the prediction.

$$\mathrm{RMS}E = \sqrt{\mathrm{MSE}}. \quad (10)$$

MAE (Mean Absolute Error) MAE evaluates the degree of deviation between the true value and the predicted value, which can accurately reflect the size of the actual prediction error.The smaller the MAE value, the better the quality of the model and the more accurate the prediction.

$$MAE = \frac{1}{n}\sum_{i=1}^{n}|\hat{y} - y_i|^2. \quad (11)$$

The model was tested for each of the Great Lakes individually to calculate the error from the optimal water level and compare it to the actual data for 2017, and the results are shown in Table 2.

Since the three indicators MSE, RMSE, and MAE are all as small as possible, analysing the data in the table, it can be obtained that the model's MSE, RMSE, and MAE values for the predicted water levels in each of the Great Lakes compared to the optimal levels are all smaller than the actual water level data of 2017, which shows that the model's control is closer to the optimal water level than the actual water level of 2017. Therefore, our new control measures can make the actual recorded water levels satisfactory for the stakeholders in that year.

Table 2. Comparison of Model Predictions with Actual Water Levels for the Great Lakes in 2017

	MSE		RMSE		MAE	
comparison	actual data	model data	actual data	model data	actual data	model data
Lake Superior	0.044551328	0.015381769	0.211071856	0.12402326	0.98427078	0.994569338
Lake Michigan	0.009967795	0.001610445	0.099838846	0.040130345	0.995220904	0.999530084
Lake St.Clair	0.007136	0.001472938	0.084475	0.04234689	0.996735	0.999325987
Lake Erie	0.008979907	0.001342979	0.094762372	0.034212792	0.994573159	0.999188396
Lake Ontario	0.04595056	0.008908702	0.214360817	0.094385923	0.969176636	0.994024095

5. Conclusion

This study addresses the water level management issues of the Great Lakes by proposing a weight indicator system based on the Analytic Hierarchy Process (AHP) and introducing Partial Differential Equations (PDEs) to simulate and control the flow and water level changes of the lakes. By applying the 3δ principle to the water level data over the past 20 years, we successfully eliminated outliers, providing an accurate data foundation for subsequent analysis. In the constructed weight indicator system, we categorized various indicators affecting water levels and determined the weights of each indicator using the AHP. This system not only considers the needs of different stakeholders such as construction, navigation, ecology, power generation, and agriculture but also adapts to the specific conditions of each lake by constructing different weight matrices. A significant contribution of the article is the application of PDEs to the independent variables of lake depth and river flow velocity, thereby establishing a comprehensive model to address water level control issues. The model not only takes into account the influence of various factors but also obtains numerical solutions through numerical integration methods, providing a new computational and predictive tool for the water level management of the Great Lakes. Sensitivity analysis further demonstrates the model's sensitivity to changes in dam flow rates, showing the stability and reliability of the model under different flow conditions. The analysis results indicate that our model can effectively meet the needs of stakeholders and provide a feasible solution for water level control in the Great Lakes.

Although this study has achieved certain results, we recognize that there is still room for improvement in the model. Future research can further explore other potential factors affecting water levels, such as climate change, extreme weather events, and human activities. In addition, the model's parameters and algorithms also need to be adjusted and optimized according to actual conditions to improve the model's adaptability and accuracy.

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