The Foundation of Goldbach’s Conjecture

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Abstract: Based on the congruence theory and the quadratic sieve method of stacking barriers, this paper is aimed to prove it using the list method and mathematical induction: when $n \geq 1$ time, Large even number $p_n^2 < 2b < p_{n+1}^2$, $b \in B_n = \{b_n \mid 0 \leq b_n \leq \pi_n'\}$ time, Different simultaneous residues $\delta_n(b_n) \neq \pm b_n \pmod{p_i}, \ i = 1, 2, \cdots, n$. About Mo $\pi_n = p_1p_2\cdots p_n$ the Solution set $\Delta_n(b_n) \pmod{\pi_n}$ is found that, there must be at least one absolute minimum special solution $0 \leq |\delta_n(b_n)|_{\min} \leq 2^{-1}(p_n^2 - p_n)$. Make $b \pm \delta_n(b_n)_{\min}$ Both are greater than $p_n$. Odd prime number. So it proves from this: Large even number $2b$ Can be expressed as a pair greater than $\sqrt{2b} > p_n$ The sum of two odd prime numbers of.

1. Main Article

Analysis: Obviously, to prove the Goldbach conjecture, it is only necessary to prove for any given sufficiently large even number $2b > 4$, Can be expressed at least as the sum of a pair of odd prime numbers. Existing $2b > 4$ Is a sufficiently large even number given in advance; because the number of odd prime numbers is infinite, which must exist $p_n^2 < 2b < p_{n+1}^2$ that Can make $\pi_n = p_1p_2\cdots p_n$ As a module, Come and seek for $2b > 4$ Two odd prime numbers of.

Solution idea one: If we can prove that when any large integer $b > 2$ time, At least one tolerance exists $\delta(b)$. Make $b \pm \delta(b)$ For a pair less than $2b$ Odd prime number $p_1, p_2$ The Goldbach conjecture holds. And by the prime judgment method: About $b$ Such tolerance $\delta(b) \geq 0$ The following two conditions must be met:

Condition one. When $p_n^2 < 2b < p_{n+1}^2$ time $b \pm \delta(b)$ Must be modular $\pi_n = p_1p_2\cdots p_n$ Simplified remainder of which is $b \pm \delta(b)$ Must be $\pi_n = p_1p_2\cdots p_n$ Mutual prime. Which is: $b \in B_n = \{b_n \mid 0 \leq b_n \leq \pi_n' = p_2p_3\cdots p_n\}$ And

$\{b_n \pm \delta_n(b_n), \ \pi_n\} = \{b_n \pm \delta_n(b_n), p_2p_3\cdots p_n\} = 1, \ \iff b_n \pm \delta(b_n) \neq 0 \pmod{p_i}, i = 1, 2, \cdots, n \iff \delta_n(b_n) \neq \pm b_i \equiv \pm r_i \pmod{p_i}, i = 1, 2, \cdots, n$.

Conditions 2: must prove that $b$ At least one absolute minimum tolerance $0 \leq |\delta_n(b)|_{\min} \leq 2^{-1}(p_n^2 - p_n) < b$, should make sure $b \pm \delta_n(b)_{\min}$ For two greater than $p_n$ And less than $2b$ Odd prime numbers.

Problem solving idea two: For any given large even number, $2b > p_n^2 \geq 4$, Prove that at least one odd prime number exist $p_n < 2b$. While ensuring $2b - p_n$ Also for prime.
2. Introduction to Stacked Sieve Grid

2.1 The generation (symbol) number used in the stacking grid sieve method.

2.1.1 Unless otherwise stated, all lowercase foreign letters in this document refer to integers, and uppercase foreign letters refer to sets.

2.1.2 For different cog symbols. Example: \( a \not\equiv b \pmod{m} \) Table about mold \( m \), \( a \) and \( b \) Difference.

2.1.3 Left subscript band "\( \pm \)" The meaning of the collection of numbers: \( \pm \{b\} \) Table absolute value \( b \geq 0 \) Of two opposite numbers. Example: \( \pm \{b\} = \{\pm b\} \).

2.1.4 \( \pi'_n = p_2 p_3 \cdots p_n \). \( \pi_n = p_1 p_2 \cdots p_n \).

2.1.5 Number of primes \( p_k \) Euler function. \( \phi(p_k) = p_k - 1 \)

2.1.6 \( \psi(p_k) \) Number of primes \( p_k \) Loyalty function, and agrees: If and only if \( k = 1 \) time:

\[ \psi(p_k) = \psi(p_1) = 1 \] when \( k \geq 2 \) times; \( \psi(p_k) = p_k - 2 \) and, \( \psi(p_1 p_2 \cdots p_k) = \psi(p_1) \psi(p_2) \cdots \psi(p_k) \) Are prime numbers that vary in pairs.)

2.1.7 Collection symbols: \( \Xi_n(b_n), \Delta_n(b_n) \) Both table and \( b_n \) \( (1 \leq b_n \leq \pi'_n) \) are some kind of collection. Fore example:

\[ \Xi_n(b_n) = \{ \{0 < \xi_n(b_n) < \pi_n\} | (\xi_n(b_n) \times [2b_n - \xi_n(b_n)], \pi_n) = 1\} \subseteq \Xi_n = \{ \{\xi_n | (\xi_n, \pi_n) = 1\}\} \]

\[ \Delta_n(b_n) = \{ 0 \leq \delta_n(b_n) < \pi_n | b_n \pm \delta_n(b_n) \in \Xi_n \} \]

2.1.8 Addition, Subtraction, and Multiplication Algorithms for Integers and Sets

\[ c \pm \{a_1, a_2, \ldots, a_k\} = \{c \pm a_1, c \pm a_2, \ldots, c \pm a_k\}, c \times \{a_1, a_2, \ldots, a_k\} = \{a_1 c, a_2 c, \ldots, a_k c\} \].

2.2 Nouns, definitions and basic concepts of sieving grid method.

2.2.1 If \( b_n + b'_n = \pi'_n \) Regular \( b_n, b'_n \) Mutually \( \pi'_n \) The reverse residue, \( b'_n \) Top mark \( \leftarrow \) for \( \pi'_n \)

Reverse residual symbol. For example: \( b_n = \pi'_n - b_n \) Inverse surplus; Empathy set \( B_n \) For collection \( B_n \) about \( \pi'_n \) Reverse Residual Set: \( B_n = \pi'_n - B_n \)

2.2.2 module \( \pi'_n \) Remaining class \( b_n \) of \( n \) (Multi) dimensional. \( \text{If } a_n \equiv r_i \pmod{p_i}, i = 1,2, \cdots, n \) Then:

definition \( \{r_1, r_2, \ldots, r_n\} \) of \( n \) (Multi) dimensional, Referred to as: \( b_n = \{r_1, r_2, \ldots, r_n\} \pmod{\pi_n} \) It is agreed that when it does not cause misunderstanding, it can be directly abbreviated as: \( b_n \equiv \{r_1, r_2, \ldots, r_n\} \) Known from Chinese Remainder Theorem and Euler's Theorem:

\[ b_n = \sum_{i=1}^{\phi(n)} \left( \frac{\pi_n}{p_i} \right)^{\phi(p_i)} \pmod{\pi_n} \] (Prove slightly!)

2.2.3 Module \( \pi_n = p_1 p_2 \cdots p_n \) Remaining system

2.2.3.1 Module \( \pi_n \) Complete residual system. Module \( \pi_n \) The minimum positive completely residual system is \( A_n, A_n = \{1, 2, \cdots, \pi_n\} \). For example: \( A_5 = \{1, 2, 3, \cdots, 30\} \). But in \( A_n \)

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There are two categories of parity, Of which odd

Not the subject of this article; And even

Is the key research object of this article, of which

Complete the residual system.

2.2.3.2 Module $\pi_n$ Simplified remainder $\tilde{\varepsilon}_n$ And simplifying the remaining system

(See 1.3.1.2 Example 1 for details). module $\pi_n$ but in $A_n$ There are two categories of parity, Of which the odd

Not the subject of this article; And even

Is the key research object of this article, among them $B_n$ for module $\pi_n = p_2 p_3 \cdots p_n$ Complete residual system?

2.2.3.3 Module $\pi_n$ Simplified remainder $\tilde{\varepsilon}_n$ And simplifying the remaining system

(See 1.3.1.2 Example 1 for details). for module $\pi_n$ The general table of simplified residuals is: 1) mold $\pi_n$ Minimal positive simplified residual system

And simplifying the remaining system.

Knowing from the prime discrimination method:

For example:

2.2.4 Module $\pi_n$ which is the Remaining system

2.2.4.1 $B_n$ the general table model $\pi_n$ Minimal complete remainder $B_n = \{b_n | 1 \leq b_n \leq \pi_n\}$

Or the table Model $\pi_n$ Minimal nonnegative complete residual system. And

Table model $\pi_n$ Minimally singular completely residual system of $B_{n, \pi} = \{b_{n, \pi} | b_{n, \pi} = 2a , 0 \leq a \leq \pi_n - 1\}$ module $\pi_n$ The smallest non-negative even complete residual system.

2.2.4.2 Module $\pi_n$ Simplified residual system $\tilde{\varepsilon}_n$ Inside and only $\phi(\pi_n)$ Model category $\pi_n'$

Simplified remainder of $\tilde{\varepsilon}_n$ the general table model $\pi_n'$ Minimal positive simplified residual system:

Old model $\pi_n'$ is the Simplified remainder $\tilde{\varepsilon}_n$ About the axis $S_n = 0 (mod \pi_n')$ Symmetrically distributed, And because $\pi_n'$ Is odd, so $\tilde{\varepsilon}_n'$ and $\tilde{\varepsilon}_n = \pi_n' - \tilde{\varepsilon}_n'$ The opposite parity, which is $\tilde{\varepsilon}_n$ and $\tilde{\varepsilon}_n' = \pi_n' - \tilde{\varepsilon}_n'$ They must be odd and even. Old model $\pi_n'$ Minimal positive simplified residual system $\tilde{\varepsilon}_n'$ Also available $\pi_n$ The table of the minimal residuals and their inverse residuals is:

For example:
\[ \Xi_3 = \{1, 7, 11, 13, 11, 7, 1\}. \]

### 2.3 Stacked lattice sieve method for different residuals and its solution set.

Lattice method uses continuous unit lattice points to represent continuous integers or modulos

\[ \begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\end{array} \]

Figure 1. grid point diagram of positive integer sequence

\( p_k (k \geq 1) \) Continuous minimal non-negative (or absolute minimum) remainder of, A (graphical) method to study the nature of integers. For example: positive integer sequence can be expressed as

\[ \begin{array}{cccccccc}
0 & 1 & 2 & -2 & -1 & 0 & 1 & 2 \\
\end{array} \]

Figure 2. The continuous absolute minimum residual lattice of a module

#### 2.3.1 Prime numbers \( p_k \) Differential Remainders of the First Degree and Prime Modulus \( p_k \) Once sieve.

#### 2.3.1.1 Defining different remainders once in a unary \( x_k(r_k) \neq r_k \pmod{p_k} \) for Prime \( p_k \) time \( r_k \) (Once) sieve, Referred to as \( s_k[r_k] \) Its sieve diagram is Figure 3:

\[ \begin{array}{cccccccc}
\vdots & 0 & 1 & 1 & 2 & \vdots & \vdots & \vdots \\
\end{array} \]

Figure 3. \( s_k[r_k] \) grid point diagram screen diagram

Of which: include 0 of the Lattice points \( p_k \) Original (lattice) point of \( r_k \) The grid point is called \( s_k[r_k] \) Of the sieve, cover \( s_k[r_k] \) Sieve and (delete) the dead one \( p_k \) the \( r_k \) After the class left for the retained model \( p_k \) \( r_k \) Take a number from each class \( s_k[r_k] \) Solution set, Referred to as: \( X_k(r_k) \) for example: \( s_3[2] \) Table one unary different remainder \( x_3 \neq 2 \pmod{5} \) Its sieve diagram is:

\[ \begin{array}{cccc}
0 & 1 & 2 & 3 \\
\end{array} \]

Figure 4. \( s_3[2] \) grid point diagram screen diagram

\( X_3(2) = \{0, 1, 3, 4\} \) represented as \( s_3[2] \)

About the module \( p_3 \) which is the Minimum non-negative solution set.

#### 2.3.1.1 If: \( 0 \leq b_n \leq \pi_n \) Then define different sets of simultaneous residuals: \( x_n(b_n) \neq r_j \pmod{p_i} \), \( i = 1, 2, \cdots, n \) Is (combined) module \( \pi_n \) Pair \( b_n \) is the Sieve, Referred to as: \( S_n[b_n] \) or \( S_n[\{r_1, r_2, \cdots, r_n\}] \) and its called \( b_n \) for \( S_n[b_n] \) Sieve "of". The minimum non-negative solution set of this simultaneous set is recorded as: \( X_n(b_n) \) which is the Solution set of \( X_n(b_n) \) which can be Stackable \( s_i[r_i] \) \( i = 1, 2, \cdots, n \) and the Combination sieve of \( S_n[b_n] \) obtain.

\[ S_n[b_n] = S_n[\{r_1, r_2, \cdots, r_n\}] \equiv \bigcup_{i=1}^{n} s_i[r_i] \equiv s_1[r_1] \cup s_2[r_2] \cdots \cup s_n[r_n] \pmod{\pi_n} \]

As a special case: when,
0 = \sum_{i=1}^{n} b_i S_i[0] = S_n[0] = \bigcup_{i=1}^{n} s_i[0]

The solution set \( X_n(0) \) which is the module of \( \pi_n \) Simplified residual system \( \Xi_n \). Example 1: Sieve once with a grid \( S_i[0] \) for Module \( \pi = 30 \). Minimal positive simplified residual system (set) \( \Xi_3 \)

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Figure 5. Drawing one \( S_i[0] = S_i[\{0,1,2,5\}] \) for Module \( \pi = 30 \). Minimal positive simplified residual system (set) \( \Xi_3 \).

From the figure:\( \Xi_3 = \{1,7,11,13,17,19,23,29\} = \{1,7,11,13\} \) (mod 30).

2.3.2 Odd prime numbers \( p_k \) is the quadratic different remainders of 1 and the graphic method "quadratic prime sieve method" of finding its solution set. If \( k \geq 2 \), then \( r'_k \equiv r''_k \) (mod \( p_k \)) Then define simultaneous different remainders: \( x \equiv 0 \) (mod \( p_k \)) for module \( p_k \) of \( r'_k \) and \( r''_k \)

The general quadratic different remainder of. And agree that the quadratic different remainder can be abbreviated as: \( x \equiv \{r'_k, r''_k\} \) (mod \( p_k \)) the Solution set is \( X \) that can be Odd prime \( p_k \) pair \( r'_k \cdot r''_k \) is the second sieve \( s_i[\{r'_k, r''_k\}] \) that is obtain, but obviously the module \( p_k \) is the complete Sieve removal of \( r'_k, r''_k \) which is the two types of surplus, From the rest \( \psi(p_k) = p_k - 2 \) Taking a number from each of the class residues, which is about the module \( p_k \) but a solution set of the different remainders of \( X \). And set \( X \) as the number of different elements within (the cardinality) \( |X| = \psi(p_k) = p_k - 2 \) For example: the Different remainder \( x \equiv 0,1 \) (mod 5) is the minimal positive solution set of \( X \). Available through mold 5 of 0,1 which is the Second sieve diagram \( s_i[0,1] \). \n
Figure 6. \( s_i[0,1] \) the second sieve

Knowing that \( X = \{2,3,4,1\} \) \( X \) is the Cardinality \( |X| = |p_3| = \psi(p_3) = 5 - 2 = 3 \) (Note: Figure \( s_i[0,1] \) middle, Numbers in red grid \( \{0,1\} \). For screened molds 5 Of the two categories remaining, \( 3 \) Class number \( \{2,3,4\} \) for the set of different remainders \( x \equiv \{0,1\} \) (mod 5) is the Solution set of \( X \).

And specifically define different remainders: \( x_k \equiv r_k \) (mod \( p_k \)) if: \( x \equiv \pm r_k \) (mod \( p_k \)) for Odd prime \( p_k \) of \( \pm r_k \). The special quadratic different remainders of, is called the Modular \( p_k \) for \( \pm r_k \) is the Second sieve \( s_i[\pm r_k] \) definition \( \pm r_k \) For the Second sieve: \( s_i[\pm r_k] \) when the "Sieve", is referred to as \( \theta(r_k) = \pm r_k \).

Because \( k \geq 2 \), time: in spite of \( r_k \equiv -r_k \) (mod \( p_k \)) but \( x^2 - r_k^2 \equiv 0 \) (mod \( p_k \)) and \( x_k^2 - (r_k)^2 \equiv 0 \) (mod \( p_k \)) Constant for the same solution equation, Therefore \( s_i[\pm r_k] \) and \( s_i[\pm (-r_k)] \) Classified
as modular $p_k$ is the same category of sieve, written as: $s_k[\pm r_k] \equiv s_k[\pm(-r_k)]$ obviously, the solution set of the same sieve is also the same: $X_k (r_k) \equiv X_k (-r_k)$.

The same definition: $\Theta(b_k) = \{\pm b_k \} \equiv \{r_k - r_k \} \equiv \{r_k \} \pmod{p_k}, 0 \leq |r_k| < 2^{-1} \phi(p_k)$.

For module $p_k$ which is the second sieve $s_k[\pm b_k]$ Set of classes and fixed $\Theta(r_k)$ is the Cardinality: when $r_k = 0 \text{ time } |\Theta(r_k)| = 1; \text{ when } r_k \neq 0, |\Theta(r_k)| = 2$.

2.4 Module $\pi_n$ Quadratic simultaneous different remainder groups $\zeta_n \equiv$ 

\[
\begin{cases}
0 \pmod{2} \\
0, 1 \pmod{p_i}, i = 2, 3, \cdots n \pmod{\pi_n}
\end{cases}
\]

is the Minimal positive solution set

Greek letters $\zeta$ is the Uppercase, it can be module $\pi_n$ is the second sieve, $s_i[0] \cup S'_i[0,1] = s_i[0] \cup s_{3,1}[0,1,1] \cup s_{3,2}[0,1,1] \cup s_{3,3}[0,1,1]$ (mod $\pi_n$) that is obtain, and from the permutations and combinations, we can know:

$Z_n = \psi(\pi_n) = \psi(\pi'_n) = \psi(p_2)\psi(p_3)\cdots\psi(p_n) = (p_2 - 2)(p_3 - 2)\cdots(p_n - 2)$.

Example 2: Using the lattice sieve method to find different sets of simultaneous residues: $\zeta_4 \equiv$

\[
\begin{cases}
0 \pmod{2} \\
0, 1 \pmod{p_i}, i = 2, 3, 4 \pmod{\pi_4}
\end{cases}
\]

is the Solution set $Z_4$ the extremely Cardinality $\|Z_4\|$

Solution: is the intent Mapping $s_i[0] \cup S'_i[0,1]$ the second sieve.

Figure 7. $s_i[0] \cup S'_i[0,1]$ the second sieve

from Figure 7 $\zeta_4 \equiv$

\[
\begin{cases}
0 \pmod{2} \\
0, 1 \pmod{p_i}, i = 2, 3, 4 \pmod{\pi_4}
\end{cases}
\]

is the Minimal positive solution set:

$Z_4 = \{17, 23, 47, 53, 59, 83, 89, 107, 137, 143, 149, 167, 173, 179, 209\}$ (mod $\pi_4$) (Point: Collection $Z_4$, Very importantly, it will be used for later proofs!) Knowing the set by permutation and combination $Z_4$ which is the Cardinality:
\[ \|Z_4\| = \psi(p_1') = \psi(p_2)\psi(p_3)\psi(p_4) = (p_2 - 2)(p_3 - 2)(p_4 - 2) = 15 \]

2.5 Module \( \pi_n \) is the Unary quadratic simultaneous different remainders \( \delta_n(b_n) \equiv \pm b_n \equiv \pm r_i \) (mod \( p_i \), \( i = 1, 2, \ldots, n \)) the Solution set and its laws, if: \( n \geq 2 \),

\[ b_n \in B_n = \{ b_n : 0 \leq b_n < \pi_n \}, \quad b_n = \{ r_1, r_2, \ldots, r_n \}, \]

Then define a group of quadratic different remainders: \( \delta_n(b_n) \equiv \pm r_i \) (mod \( p_i \), \( i = 1, 2, \ldots, n \)) for Module, \( \pi_n \) of \( \pm b_n \) is the Special one-variable quadratic different remainder groups, the Solution set of \( \Delta_n(b_n) \) cause the module \( \pi_n \) of \( \pm b_n \) is the heap barrier (combination) of the second sieve:

\[ S_n[\pm b_n] = s_1[\pm r_1] \cup s_2[\pm r_2] \cdots \cup s_n[\pm r_n] = \bigcup_{i=1}^{n} s_i[\pm r_i] \]

which is to get the sieve diagram.

Cause at 1.3.2 when realized:

types of secondary sieve in the primary secondary sieve system: \( s_i[0] \) and \( s_i[\pm 1] \); when \( k \geq 2 \) time:

Because the odd prime \( P_k \) of \( \pm r_k \) is the primary secondary sieve system. \( s_i[\pm r_i] \) There is only \( 2^{i-1}(P_k + 1) \) Class two: second sieve \( s_i[0] \), \( s_i[\pm 1] \),

\[ \cdots s_k[\pm 2^{i-1}(P_k)] \]

just as \( r_k = 0 \) is \( s_i[0] \) There is only one type of sieve; and just as \( r_k \neq 0 \) is \( s_i[\pm r_k] \) There are two similar sieves in it: \( s_i[\pm r_k] \equiv s_i[\pm(-r_k)] \). The set of similar sieve is \( \Theta(r_k) = \{ r_k, -r_k \} \). \[ \text{when } r_k = 0, \quad \|\Theta(r_k)\| = 1; \quad \text{when } r_k \neq 0, \quad \|\Theta(r_k)\| = 2. \]

Main point:

\[ 1 \leq (b_n, \pi_n') = G_n = g_1g_2\cdots g_n = \pi_n', \quad 1 \leq L_n = \pi_n'/G_n = l_1l_2\cdots l_v \leq \pi_n', \]

g_1, g_2, \ldots, g_n, l_1, l_2, \ldots, l_v two different odd prime numbers.) From the permutation and combination:

module \( \pi_n \) for \( \pm B_n \) is the sieve \( S_n[\pm B_n] \) it has only Inside \( 2^{\pi_n - 1} \)

\[ S_n[\pm\{0, 0, \ldots, 0\} \}, S_n[\pm\{0, 0, \ldots, 1\} \], \]

Class two of two different sieves:

\[ S_n[\{0, 0, \ldots, 2\} \}, \ldots, S_n[\{1, 2, \ldots, 2^{-1}\phi(p_n)\} \] cause \[ \cdots S_n[\{1, 2, \ldots, 2^{-1}\phi(p_n)\} \] There have and the only sieve \( 2^v \geq 1 \) which are the Kind of sieve, which is the Collection of sieves \( \Theta_n(b_n) \) is the Cardinality \( \|\Theta_n(b_n)\| = 2^v \), And its solution set, \( \Delta_n(b_n) \) is the Cardinality \( \|\Delta_n(b_n)\| = \phi(G_n)\psi(L_n) \). From this if \( G_n \) The greater the number of prime factors, the smaller the value of the prime factors, then: the Solution set \( \Delta_n(b_n) \) which is the Cardinality \( \|\Delta_n(b_n)\| = \phi(G_n)\psi(L_n) \). It follows that if: \( G_n \) The greater the number of prime factors, the smaller the value of the prime factors, then: is the Solution set \( \Delta_n(b_n) \) the cordiality \( \|\Delta_n(b_n)\| \) it greater, while:

\[ S_n[\pm b_n] \] The number of the same type of sieve in the sieve class is the set of this type of sieve \( \Theta_n(b_n) \) is the cordiality of \( \|\Theta_n(b_n)\| = 2^v \), it the more smaller. Because \( b = b_n + t\pi_n \equiv b_n \) (mod \( \pi_n \)), \( t \in Z \) \( S_n[\pm b_n] \equiv S_n[\pm(-b_n)] \) so is the mole. \( \pi_n : \Delta_n(b_n) \) as there are two important properties: Nature one
\[ \Delta_n(b_n + t\pi_n) = \Delta_n(b_n); \] Nature 2 \( \Delta_n(-b_n) = \Delta_n(b_n) \) now \( n = 4 \) the Sieve system \( \mathcal{S}_4 \) \((b_n \subset B_n)\) as well as the System of solutions \( \Delta_n(b_n) \) \((b_n \subset B_n)\) that is Listed in Figure 8 (see next page) for reference. For query: \( \Theta_4(b_n), \Xi \Delta_4(b_n), \| \Theta_4(b_n), \| \Delta_4(b_n) \) and so on.

<table>
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<tr>
<th>( n )</th>
<th>( \Theta_4(b_n) )</th>
<th>( \Xi_4 \Delta_4(b_n) )</th>
<th>( | \Theta_4(b_n), | \Delta_4(b_n) )</th>
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<td>5</td>
<td>[ 2, 3, 5, 7 ]</td>
<td>[ 2, 3, 5, 7 ]</td>
<td>[ 2, 3, 5, 7 ]</td>
</tr>
</tbody>
</table>

Figure 8. Table of \( \Delta_n(b_n) = \{ \delta_4(b_n) \} \) \((\text{mod } p, \ n = 1, 2, \ldots, n)\)

Example 3: From Figure 8, we can find out:

\[ \Theta_4(1) = \{ 1, 29, 41, 71, 139, 169, 181, 209 \} \] \((\text{mod } p_4)\),

\[ \Xi_4 \Delta_4(29) = \cdots \Xi_n \Delta_4(209) = \{ 0, 12, 18, 30, 42, 60, 72, 102 \}, \]

\[ \| \Theta_4(1), \| \Delta_4 \] \((\text{mod } p_4)\)

And:

\( \theta_4(1) \in \Theta_4(1), \delta_4(1) \in \Delta_4 \)

Then we know:

2.5.1. \( \theta_4(1) \pm \delta_4(1) \) is and for \( \theta_4(1) \) Two unequal modules \( \pi_4 = 210 \) The simplified residual sum

of \( 1 < \theta_4(1) \pm \delta_4(1) < p_4 = 121 \theta_4(1) \pm \delta_4(1) \) That is and for \( \theta_4(1) \) Odd unequal primes of. For example:

\( 29, 12 = \{ 17, 41 \} \subset P \), \( 29, 12 = \{ 11, 47 \} \subset P \), \( 41, 12 = \{ 29, 53 \} \subset P \), \( 41, 12 = \{ 29, 53 \} \subset P \), \( 41, 30 = \{ 11, 71 \} \subset P \).

2.5.2. \( 71, 42 = \{ 39, 113 \} \subset P \) the Same difference \( \theta_4(1) \) Two unequal modules \( \pi_4 = 210 \) Simplified remainder of \( \delta_4(1) \pm \theta_4(1) \in \Xi_4 \), And when \( 1 < \theta_4(1) \pm \delta_4(1) < p_4 = 121 \) time:

\( \theta_4(1) \pm \delta_4(1) \) That is, the phase difference is \( \theta_4(1) \) Odd odd prime numbers of. Especially when taking \( \theta_4(1) = 1, \delta_4(1) \in \Delta_4(1) = \{ 0, 12, 18, 30, 42, 60, 72, 102 \}, 1 < \delta_4 \pm \theta_4(1) < p_4 \) time,

\( \delta_4(1) \pm \theta_4(1) = \delta_4(1) \pm 1 \) For twin primes: \( 12 = \{ 11, 13 \}, 18 = \{ 17, 19 \} \).
30 ± 1 = \{29, 31, \ldots \}; \quad 108 ± 1 = \{107, 199, \ldots \}

(Note: This phenomenon suggests that the use of stacking sieve method can further prove the infinity of twin primes.)

2.5.3. Lemma part: \( n \geq 2 \); \( \xi_n' [\gamma] \in \Xi_n = \{1 \leq \xi_n < \pi_n' \mid (\xi_n, \pi_n') = 1\} \),

[Lemma 1] If:

\[ \Xi_n = \{1 \leq \xi_n < \pi_n \mid (\xi_n, \pi_n') = 1\} \]

\[ \Xi_n (\xi_n' [\gamma]) = \{1 \leq \xi_n (\xi_n' [\gamma]) = \xi_n (\xi_n' [\gamma]) \in \Xi_n, \quad 2 \xi_n' [\gamma] - \xi_n (\xi_n' [\gamma]) \in \Xi_n, \subset \Xi_n, \}

\[ Z_n = \{\xi_n (\xi_n', -1), \quad \pi_n' = 1\} \]

\[ \Xi_n (\xi_n' [\gamma]) = \{\xi_n (\xi_n' [\gamma]) \mid \xi_n (\xi_n' [\gamma]) \equiv (2 \xi_n' [\gamma] - \pi_n') \zeta_n \pmod{\pi_n}, \zeta_n \in Z_n, \subset \Xi_n. \]

Then:

Order: \( x_n \in X_n = \{x_n \mid (x_n, (x_n - 1), \pi_n') = 1\} \); \( y_n \in Y_n = \{y_n \mid (y_n, 2) = 1\} \)

\[ X_n \cap Y_n = Z_n = \{\xi_n (\xi_n', -1), \quad \pi_n' = 1\} \quad \Xi_n = \psi (\pi_n) \times \text{Causal conditions and definitions:} \]

\( (\xi_n' [\gamma], \pi_n') = 1, \quad (x_n, \pi_n') = 1, \quad \text{if: } (2 \xi_n' [\gamma] x_n - \pi_n') = 1 \), therefore:

\( (2 \xi_n' [\gamma] x_n - \pi_n') = 1, \quad \text{Then: } 2 \xi_n' [\gamma] x_n - \pi_n' \in \Xi_n, \quad \text{present order: } \xi_n (\alpha) \equiv 2 \xi_n' [\gamma] x_n - \pi_n' \pmod{\pi_n} \) \((1) \)

The same can be proved: \( \xi_n (\beta) \equiv \pi_n' - 2 \xi_n' [\gamma] (x_n - 1) \in \Xi_n \pmod{\pi_n} \) \((2) \)

From \((1); (2): \xi_n (\alpha) + \xi_n (\beta) \equiv 2 \xi_n' [\gamma] (\mod{\pi_n}) \).

Knowing that: \( \xi_n (\alpha) \) which is requested for: \( \xi_n (\xi_n' [\gamma]) \).

Now in \((1); x_n \) replaced with \( \zeta_n \in Z_n = X_n \cap Y_n \) then can get: \( \xi_n (\xi_n' [\gamma]) \equiv (2 \xi_n' [\gamma] - \pi_n') \zeta_n \pmod{\pi_n} \).

Which is:

\[ \Xi_n (\xi_n' [\gamma]) = \{\xi_n (\xi_n' [\gamma]) \mid \xi_n (\xi_n' [\gamma]) \equiv (2 \xi_n' [\gamma] - \pi_n') \zeta_n \pmod{\pi_n}, \zeta_n \in Z_n, \subset \Xi_n. \]

(Lemma 1 is proven.)

[Lemma 2] If:

\( n \geq 2, 1 \leq b_n \leq \pi_n', p_2 \leq G_n = (b_n, \pi_n') \leq \pi_n' \pi_n' = L_n G_n \leq L = \pi_n' / G_n \leq \pi_n' / p_2, \]

\( C_n = \{c_n (c_n, G_n) = 1\} = \{c_n [1], c_n [2], \ldots c_n [\phi (G_n)]\} \); \( b_n + L_n c_n [i] \equiv \xi_n' [\gamma] \in \Xi_n \)

(\( \text{mod } \pi_n \), \( c_n [i] \in C_n = \{c_n [1], c_n [2], \ldots c_n [\phi (G_n)]\} \).

Approval of cause and condition \((b_n, \pi_n') = G_n \text{ Decree: } b_n = m_n G_n \) because \((b_n, \pi_n') = (m_n G_n, L_n G_n) = G_n \text{ therefore: } (b_n, L_n) = (m_n G_n, L_n) = 1, \)

And know from the conditions: \((c_n, G_n) = 1, \)

Therefore:

\( b_n + c_n L_n, G_n = (m_n G_n + c_n L_n, G_n) = (c_n L_n, G_n) = 1 \) \((1) \)

\( b_n + c_n L_n, L_n = (m_n G_n + c_n L_n, L_n) = (m_n G_n, L_n) = 1, \) \((2) \)

Therefore, \((1); (2) \) is known as \( b_n \pm c_n G_n, L_n G_n = (m_n G_n \pm c_n L_n, \pi_n') = 1 \)

But by definition: \( b_n + L_n c_n [i] \equiv \xi_n' [\gamma] \in \Xi_n \text{ (mod } \pi_n \), \( c_n [i] \in C_n \) \text{(Lemma 2 is certified.)}

[Lemma 3] \( \Delta_n (b_n) \equiv \Delta_n (b_n) \equiv \pi_n' \Delta_n (b_n) \text{ (mod } \pi_n \), \( b_n \in B_n = \{b_n \mid 1 \leq b_n \leq \pi_n \} \).

Order of: \( \delta_n (b_n) \mid 2 \equiv \delta_n (b_n) \pm b_n \equiv \pm b_n \text{ (mod } \pi_n \), \( i = 1, 2, \ldots n \) \text{ (mod } \pi_n \).
\[ \Delta_n(b_n) = \left\{ \begin{array}{ll} \delta_n(b_n) & \neq \pm b_n \equiv \left\{ \begin{array}{ll} 1 \pmod{2}, & \\
 + b_n \pmod{\pi_n} & \end{array} \right. \\
 \end{array} \right. \]

Therefore:

\[ \Delta_n(b_n) = \left\{ \begin{array}{ll} \delta_n(b_n) & \neq \pm b_n \equiv \left\{ \begin{array}{ll} 1 \pmod{2}, & \\
 + b_n \pmod{\pi_n} & \end{array} \right. \\
 \end{array} \right. \]

By definition:

\[ \pm b_n \equiv \pm r_i \pmod{p_i}, i = 1, 2, \cdots, n \] (mod \( \pi_n \)) because \( b_n \equiv 1 \pmod{2} \) therefore

\[ \hat{b}_n = \pi' - b_n = \pi' - 1 \equiv 0 \]

Therefore:

\[ \Delta_n(\hat{b}_n) = \left\{ \begin{array}{ll} \delta_n(\hat{b}_n) & \neq \delta_n(\hat{b}_n) \equiv \left\{ \begin{array}{ll} 1 \pmod{2}, & \\
 + \delta_n(\hat{b}_n) & \end{array} \right. \\
 \end{array} \right. \]

Because

\[ S'_n[\pm b_n] = \bigcup_{i=2}^{n} S_u[\pm b_n] = \bigcup_{i=2}^{n} S_v[\pm (r_i)] = \bigcup_{i=2}^{n} S_u[\pm r_i] = S'_n[\pm b_n] \pmod{\pi_n} \]

Therefore:

\[ \Delta'_n(b'_n) \equiv \Delta'_n(b'_n) \pmod{\pi_n} \]

As we know that:

\[ \delta_n(b_n) = \left\{ \begin{array}{ll} 1 \pmod{2}, & \\
 + \delta_n(b_n) & \end{array} \right. \]

Because

\[ \Delta'_n(b'_n) = \Delta'_n(b'_n) \equiv \left\{ \begin{array}{ll} 1 \pmod{2}, & \\
 + \delta_n(b_n) & \end{array} \right. \]

Reasons:

\[ | - \delta_n(b_n) | = \delta_n(b_n) \pmod{\pi_n}, \]

therefore

\[ \Delta_n(b_n) \equiv \Delta_n(b_n) \pmod{\pi_n} \]

(Lemma 3I is certified.)

From Lemma 3, it is easy to know Reasoning 1, Reasoning 2.

[Inference one]

\[ \Delta_n(\xi'_{n}) \equiv \Delta_n(\xi'_{n}) \pmod{\pi_n} \]

[Inference two]

\[ \hat{\delta_n}(b_n) = \hat{\delta_n}(b_n) \equiv \pi' - \delta_n(b_n) \max; \]

\[ \delta_n(b_n) \max = \delta_n(b_n) \min = \pi' - \delta_n(b_n) \min; \]

[Lemma 4] \[ \Delta_n(\xi_n) = \xi_n \times \Delta_n(1) \pmod{\pi_n} \]

Order: \[ 0 \pmod{p_i} \] i = 1, 2, \cdots, n. Period \[ i = 1, 2, \cdots, n \].

In mold \( P_i \)

The minimum absolute value of \( \pm r_i = \pm 1 \times r_i \), \( \cdots, s_i[\pm r_i] = r_i \times s_i[\pm 1] \pmod{p_i} \)

\[ \xi_n, \pi_n = (\xi_n, p_1 p_2 \cdots p_n) = 1 \]

\[ S_n[\pm \xi_n] = S_n[\pm 1] \pmod{\pi_n} \]

(Lemma 4 is certified.) From Lemma 3 and Lemma 4, we can get:

[Lemma 5] \[ \Delta_n(\xi_n) \equiv \xi_n \times \Delta_n(1) \equiv \xi_n \times \Delta_n(1) \pmod{\pi_n} \] (Prove slightly!)
Example 4 Known as: \( \pm \Delta_4(1) = \{0,12,18,30,42,60,72,102\} \), or \( \pm \Delta_4(94) \).

Solution: cause \( (94, \pi_4) = (94, 210) = 2 \), therefore \( 94 \in \Xi_4 \), because \( \pi_4 - 94 = 11 \in \Xi_4 \).

Knowing that; \( 94 = 11 \). Therefore, by the lemma Wuzhi:

\[
\pm \Delta_4(94) = \pm \Delta_4(11) = \{0,12,30,48,72,78,90\} = \{15,27,33,57,63,75,93,105\} \quad (\text{mod } \pi_4).
\]

[Lemma 6] If \( (b_n, \pi_n') = G_n = g_1g_2 \cdots g_n, g_1, g_2, \cdots, g_n \) for \( G_n \) is \( u \) prime factors, \( L_n = \pi_n'/G_n \), \( L_n = l_1 \cdots l_i \), \( l_i \) for \( L_n \) is \( v \) prime factors,

\[ \Theta_n(b_n) = \{ \begin{cases} \theta_n(b_n) \mid \theta_n(b_n) \equiv \pm b_n \equiv \pm r_i \quad (\text{mod } p_i, \quad 0 \leq r_i \leq p_i, \quad i = 1,2,\ldots,n) \end{cases} \} \]

then

\[ \| \Theta_n(b_n) \| = 2^v. \]

Order:

\[
\| \Theta(G_n) \| = \| \Theta(g_1) \| \times \| \Theta(g_2) \| \times \cdots \times \| \Theta(g_n) \| = 1. \quad \text{when } \quad b_n \equiv L_n \equiv r_j \neq 0 \quad \text{period}
\]

\[ \| \Theta(L_n) \| = \| \Theta(l_1) \| \times \| \Theta(l_2) \| \times \cdots \times \| \Theta(l_i) \| = 2^v. \]

\[ \vdots \quad \| \Theta_n(b_n) \| = \| \Theta_n(L_nG_n) \| = \| \Theta_n(G_n) \| \times \| \Theta_n(L_n) \| = 2^v. \]

[Lemma 7] If you know:

\[ \Theta_n(1), \quad \text{when } \quad \Theta_n(\xi_n) \equiv \xi_n[\alpha] \times \Theta_n(1) \quad (\text{mod } \pi_n), \]

\[ \Theta_n(\xi_n) \equiv \xi_n[\alpha] \times \Theta_n(1) = \{ \theta_n(\xi_n[\alpha]) [1], \theta_n(\xi_n[\alpha]) [2], \cdots, \theta_n(\xi_n[\alpha]) [2^{n-1}] \} \subset \Xi_n \quad (\text{mod } \pi_n), \]

Warrant:

\[ r_1^* \text{ for mold } \pi_n \text{ Absolute minimum residue of } \]

\[ \Theta_n(1) = \{ \theta_n(1) [1], \theta_n(1) [2], \cdots, \theta_n(1) [2^{n-1}] \} \equiv \Theta_n'(1) = \{ \theta_n'(1) [1], \theta_n'(1) [2], \cdots, \theta_n'(1) [2^{n-1}] \} \subset \Xi_n \quad (\text{mod } \pi_n), \]

Reasons:

\[ \Theta_n(1) \text{ The elements in each other are mutually prime, so } \]

\[ \Theta_n(\xi_n) \equiv \xi_n[\alpha] \times \Theta_n(1) \quad (\text{mod } \pi_n). \]

Example 5: If known \( \{1,11,13\} \subset \Xi_4 \). So by the Lemma Seven Known as:

\[ \Theta_4(11) = 11 \times \Theta_4(1) = 11 \times \{1,29,41,71,139,169,181,209\} \equiv \{11,31,59,101,109,151,179,199\} \quad (\text{mod } \pi_4). \]

\[ \Theta_4(13) = 13 \times \Theta_4(1) = 13 \times \{1,29,41,71,139,169,181,209\} \equiv \{13,43,83,97,113,127,167,197\} \quad (\text{mod } \pi_4). \]

Theorem

[Theorem 1] If:

\[ n \geq 2, \quad b_n \in B_n = \{ b_n \mid 1 \leq b_n \leq \pi_n \}, \quad p_2 \leq G_n = (b_n, \pi_n') \leq \pi_n', \quad b_n = m_nG_n, \]

\[ \pi_n' = L_nG_n, \quad 1 \leq L_n = \pi_n'/G_n \leq \pi_n'/p_2 \]

\[ C_n = \{ c_n \mid c_n(G_n) = 1 \} = \{ c_n[1], c_n[2], \cdots, c_n[\phi(G_n)] \}, \quad b_n + L_n c_n[i] \equiv \xi_n[\gamma_i] \in \Xi_n \quad (\text{mod } \pi_n), \]

\[ \{ c_n[i] \in C_n \}. \]

(From Lemma 2)

\[ \Xi_n(\xi_n[\gamma_i]) = \{ \begin{cases} 1 \leq \xi_n(b_n) < \pi_n \mid \xi_n(b_n) \in \Xi_n \quad \text{when } \quad 2b_n - \xi_n(b_n) \in \Xi_n \} \}
\]

\[ \Xi_n(\xi_n'[\gamma_i]) = \{ \begin{cases} 1 \leq \xi_n(\xi_n'[\gamma_i]) < \pi_n \mid \xi_n(\xi_n'[\gamma_i]) \in \Xi_n \quad 2b_n - \xi_n(\xi_n'[\gamma_i]) \in \Xi_n \}
\]

\[ \text{when } \Xi_n(b_n) = \sum_{i=1}^{\phi(G_n)} \Xi_n(\xi_n'[\gamma_i]), \quad \Xi_n(b_n) = \phi(G_n)\psi(L_n). \]

Order:

\[ x_n \in X_n = \{ x_n \mid (x_n(x_n - 1), L_n) = 1 \} = \{ x_n[1], x_n[2], \cdots, x_n[\psi(L_n)] \}. \]
\[ \|X_n\| = \psi(L_n) \geq 1, \ y_n \in Y_n = \{ y_n | (y_n, 2G_n) = 1 = (y_n[1], y_n[2], \cdots y_n[\phi(G_n)]) \}, \]
\[ \|Y_n\| = \phi(G_n) \geq 1, \ X_n \cap Y_n = W_n = \left\{ \begin{array}{l}
(\psi_n (\zeta_n - 1), \ \pi_n') = 1 \\
(\zeta_n, \ 2) = 1
\end{array} \right\} \]
\[ Z_n = \left\{ \begin{array}{l}
(\zeta_n (\zeta_n - 1), \ \pi_n') = 1 \\
(\zeta_n, \ 2) = 1
\end{array} \right\}. \]

\[ \Rightarrow \text{By order of } (2, L_n) = (m_n G_n, L_n) = (x_n, L_n) = 1, \]
\[ \Rightarrow (2m_n G_n x_n - L_n y_n, L_n) = (2m_n G_n x_n, L_n) = 1, \]------- (1) \[ \Rightarrow (2m_n G_n x_n - L_n y_n, G_n) = (L_n y_n, G_n) = 1, \]------- (2)

Knowing from (1) (2): \[ (2m_n G_n x_n - L_n y_n, L_n G_n) = (2m_n G_n x_n, \pi_n) = 1, \] which is \[ 2m_n G_n x_n - L_n y_n = \xi_n[\alpha] \in Z_n \]------ (3)

In the same way, it can be proved by order: \[ L_n y_n - 2m_n G_n (x_n - 1) = \xi_n[\beta] = \Xi_n (\mod \pi_n), \]

------ (4)

By \[ (3) + (4) \] you will know that \[ \xi_n[\alpha] + \xi_n[\beta] = 2b_n (\mod \pi_n), \] Therefore, by definition:

\[ \xi_n[\alpha] \in \Xi_n (b_n), \text{ Reason} \]
\[ \Xi_n (b_n) = \{ \xi_n(b_n) | \xi_n(b_n) = 2m_n G_n x_n - L_n y_n (\mod \pi_n), x_n \in X_n, y_n \in Y_n \}. \]------ (5)

Knowing from the conditions: \[ \Rightarrow (c_n[i], G_n) = 1 (2, G_n) = 1 \]
\[ \Rightarrow (2c_n[i] - G_n, G_n) = (2c_n[i], G_n) = 1, \] for \( (2c_n[i] - G_n, 2) = (G_n, 2) = 1 \)
\[ \Rightarrow (2c_n[i] - G_n) y_n = y_n' \in Y_n (\mod G_n) \]------ (6)

And by (5) we know that \[ \xi_n(b_n) = 2m_n G_n x_n - L_n y_n (\mod \pi_n) \] \( x_n \in X_n, \ y_n \in Y_n \)------ (7)

Put (7) into \( m_n G_n \) Replaced with \( c_n b_n \); \( x_n, y_n \) Replaced with \( w_n \) can obtain

\[ \xi_n(b_n) = (2b_n - L_n) w_n (\mod \pi_n), \ w_n \in W_n, \text{ knowing that} \]
\[ \Xi_n (b_n) = \{ \xi_n(b_n) | \xi_n(b_n) = (2b_n - L_n c_n[i]) w_n (\mod \pi_n), \ w_n \in W_n \}. \]------ (8)

And by (6) you know \( y_n' = (2c_n[i] + G_n) y_n (\mod G_n) \) Replacement (7) middle \( y_n' \) Again (7) where

\[ x_n, y_n' \] Replaced with \( w_n \), About the module \( \pi_n \)
\[ \xi_n(b_n) = 2b_n x_n - L_n y_n = 2b_n x_n - L_n (2c_n[i] - G_n) = (2b_n - L_n c_n[i] - L_n G_n) w_n = [2(b_n - L_n c_n[i]) + \pi_n'] w_n, \]

Knowing from the conditions: \( b_n + L_n c_n[i] = m_n G_n \] \( L_n c_n[i] = \xi_n[i] (\mod \pi_n), \ c_n \in C_n \) then I know:
\[ \xi_n(b_n) = (2b_n - L_n c_n[i]) + \pi_n' w_n = (2\xi_n[i] (\mod \pi_n)) \]
\[ \xi_n(b_n) = [2b_n - L_n c_n[i]) + \pi_n'] w_n = (2\xi_n[i] (\mod \pi_n)) \]------ (9)

Cause Lemma: \( \Xi_n (\xi_n[i]) = \left\{ \xi_n (\xi_n[i]) \right\} \] \( \xi_n (\xi_n[i]) = (2\xi_n[i] - \pi_n') \xi_n (\mod \pi_n), \xi_n \in Z_n. \}

And because \( w_n \supset Z_n \), Therefore, \( (9) \) Knowing: \( \Xi_n (b_n) \supset \Xi_n (\xi_n[i]), \) If
\[ \Xi_n (b_n) = \bigcup_{i=1}^{\phi(G_n)} \Xi_n (\xi_n[i]), \quad \xi_n[i] = (b_n + L_n c_n[i]) (mod G_n), \ c_n[i] \in C_n. \]

And know from the permutations and combinations:

\[ \Xi_n (b_n) = \phi(G_n) \psi(L_n). \]

(Theorem 1 is proved.)

Example 6. Verification \( n = 4, b_4 = 18 \) When the theorem is true.

Order: when \( n = 4, b_4 = 18 \) period \( \pi_n = 105, \pi_4 = 210, b_4 = 18, G_4 = (18, \pi_4) = 3, \)
By definition:
$$w_n = W_n = \begin{cases} \left\lfloor \frac{w_4(w_n - 1)}{5 \times 7} \right\rfloor = 1, \\ \left\lfloor \frac{w_4(w_n - 1)}{2 \times 3} \right\rfloor = 1, \end{cases}$$
If $$w_4 \equiv 0 \pmod{2}, 0 \pmod{3}, 0 \pmod{5}, 0 \pmod{7}.$$

Known from Figure 9:
$$W_a = \{13, 17, 19, 23, 37, 47, 53, 59, 67, 73, 79, 83, 89, 97, 103, 107, 109, \ldots\} \subset \mathbb{Z}_4 \pmod{210},$$

Knowing from the theorem:
$$\xi_n(\alpha) = \xi_n(b_n) = f(b_n, w_n) = (2b_n - L_n)w_n \pmod{\pi_n}, w_n \in W_n.$$

Know:
$$\xi_n(18) = f(18, w_4) = (36 - 35)w_4 \equiv w_4 \in W_4 = \mathbb{Z}_4(18) \subset \mathbb{Z}_4 \pmod{4}.$$------ (1)

From table one: $$\Delta_4(53) = \{0, 6, 30, 36, 54, 84, 90, 96\}, \Delta_4(88) = \{9, 15, 21, 51, 69, 75, 79, 99, 105\},$$
$$\Xi_4(53) = \{1 \leq \xi_4(53) < \pi_4 | \xi_4(53) \equiv 53 \pm \delta_4(53) \pmod{\pi_4}, \delta_4(53) \in \Delta_4(53)\}$$
$$= \{53, 47, 23, 17, 209, 179, 173, 167\}, \quad ---- (2)$$

$$\Xi_4(88) = \{1 \leq \xi_4(88) < \pi_4 | \xi_4(53) \equiv 88 \pm \delta_4(88) \pmod{\pi_4}, \delta_4(88) \in \Delta_4(88)\}.$$

$$= \{79, 73, 67, 37, 19, 13, 199, 193\}.$$------ (3)

By (1) (2) (3): $$\Xi_4(18) = W_4 = \mathbb{Z}_4[35] \cup \Xi_4[88].$$

And know from the permutations and combinations:
$$\|\Xi_4(b_4)\| = \|\Xi_4(18)\| = \phi(G_4)\psi(L_4) = \phi(3)\psi(35) = (3 - 1)(5 - 2)(7 - 2) = 30.$$

Therefore, the verification theorem is valid. (To facilitate the test of Theorem 1, $$n = 4$$ period
$$\Xi_4(b_4) (b_4 \in B_4 = \{b_4 | 1 \leq b_4 \leq \pi_4 = 105\})$$, according to $$G_4$$ The order of the values is listed as...
\[ 2^{n-1} = 2^3 = 8 \] A series of tables consisting of three tables is placed after this article for reference.

[Theorem 2] If \( n \in \mathbb{N} \), \( b_n \in B_n = \{ b_n \mid 1 \leq b_n \leq \pi_n \} \), therefore

\[
0 \leq |\delta_n(b_n)|_{\min} \leq 2^{-1}(p_n^2 - p_n).
\]

2.5.4. When \( n = 3 \) time \( p_n = p_3 = \xi_3[2] = \xi_3[3] = 5 \),

\[
\begin{align*}
\pi_n &= \pi_3 = p_3 \pi_3 = p_3 \xi_3[2] = p_3 \xi_3[3] = 2 \times 3 \times 5 = 30, \\
\Xi_n &= \Xi_3 = \{ 1 \leq \xi_3 < \pi_n \mid (\xi_3, \pi_n) = 1 \} = \{ \xi_3[1] = 1, \xi_3[2] = 7, \xi_3[3] = 11, \xi_3[4] = 13, \ldots, \xi_3[\phi(\pi_3)] = \xi_3[8] = 29 \} \\
\Delta_n(b_n) &= \Delta_3(b_3) = \{ 0 \leq \delta_3(b_3) \leq \pi'_3 \mid \delta_3(b_3) \Delta_n(b_n) = \Delta_3(b_3) = \{ 0 \leq \delta_3(b_3) \leq \pi'_3 \} \delta_3(b_3) \}
\end{align*}
\]

For \( (b_3, \pi'_3) = G_3, 1 \leq L_n = \pi'_3/G_3 < \pi'_3, L_3 = l_3 \ldots l_3 \). You have \( v_3 \) odd prime factors. Sieve map \( S_3[\pm b_3] = s_1[b_3] \cup s_2[\pm b_3] \cup s_3[\pm b_3] \equiv s_1[r_3] \cup s_2[r_3] \cup s_3[r_3] \equiv (\bmod \pi_3) \),

\[
\begin{array}{c|c|c|c|c|c}
| b_3 | s_1 | s_2 | s_3 | \Delta_3(b_3) | \\
\hline
0 & 0 & 0 & (1. 7, 11, 13, 17, 19, 23, 29.) \\
1 & 0 & 1 & (6, 7, 11, 13, 17, 23, 26.) \\
0 & 2 & 12 & (6, 9, 21, 27.) \\
1 & 2 & 8 & (9, 13, 21.) \\
0 & 0 & 15 & (2, 3, 4, 8, 14, 16, 22, 26, 28, 1) \\
1 & 1 & 9 & (2, 4, 8, 10, 20, 22, 28.) \\
0 & 3 & 2 & (4, 10, 14, 16, 22, 26.) \\
1 & 1 & 5 & (6, 12, 18, 24.) \\
1 & 1 & 11 & (0, 12, 18, 24.) \\
1 & 2 & 7, 13, (0, 6, 24.) \\
\end{array}
\]

\[ \Delta_3(b_3) = \{ \delta_3(b_3) \mid \delta_3(b_3) \neq \pm b_3 \bmod (p_i, i = 1,2,3) \} \]

Figure 10. Table of

By 1.5 mold \( \pi_n \) of \( \pm B_n \) Sieve \( S_n[\pm B_n] \) Know the law:

(1) Mold \( \pi_n \) of \( \pm B_3 \) Sieve \( S_n[\pm B_3] \). Inside and only

\[
2^{-a} \prod_{i=2}^{n} \left( p_i + 1 \right) = 2^{-3} \prod_{i=2}^{3} \left( p_i + 1 \right) = 12
\]

Class of sieve:

\[
S_n[\pm \{0_1, 0_2, 0_3\}], S_n[\pm \{0_1, 0_2, 1_3\}], S_n[\pm \{0_1, 0_2, 2_3\}], \\
\cdots S_n[\pm \{1_1, 1_2, 2_3, \phi(p_3)\}, \ldots] \text{ Old model } \pi_n \text{ of } \pm B_3 \text{ Solution set } \Delta_3(B_3) \text{ Inside and only }
\]

\[
2^{-a} \prod_{i=2}^{n} \left( p_i + 1 \right) = 2^{-3} \prod_{i=2}^{3} \left( p_i + 1 \right) = 12
\]

Class-wise two different solution sets;

\[
\Delta_3(b_3) = \{ 0 \leq \delta_3(b_3) \leq \pi'_3 \mid \delta_3(b_3) \neq \pm b_3 \equiv \pm r_i \bmod (p_i), i = 1,2,3 \} \text{ If } \delta_3(b_3) \equiv \pm r'_i \neq \pm r_i \bmod (p_i), i = 1,2,3 \}
\]

Therefore, from the Chinese remainder theorem and Euler's theorem:

\[
\delta_3(b_3) \equiv \sum_{i=1}^{3} r'_i(\pi_3/p_i)^{\phi(p_i)} \bmod (\pi_3), \quad \|\Delta_3(b_3)\| = \phi(G_3) \psi(L_3).
\]

while

(2) Because, \( L_3 \) you have \( v_3 \) odd prime factors, so \( S_3[\pm b_3] \). There are and only sieve \( 2^{v_3} \geq 1 \)

Which is a Kind of sieve, i.e. \( S_3[\pm b_3] \). Set of sieve class \( \Theta_3(b_n) = \Theta_3(b_3) \). The base is \( \Theta_3(b_3) = 2^{v_3} \geq 1 \). (4) The Seven Knowledges by Lemma, when \( b_n = b_3 = \xi_3 = \Xi_3 \), time \( \Theta_3(\xi_3) = \xi_3 \times \Theta_3(1) \bmod (\pi_3) \).
Knowing from the theorem that, if: $n \geq 2$, then $\Xi_n(b_n) = \Xi_n(b_3) = \bigcup_{i=1}^{\phi(G_i)} \Xi_n(\xi_i[\gamma_i])$.

And by 1.5 module (mod $p_i$, $i = 1, 2, \cdots, n$). The rule of solution set is known as the, solution set $\Delta_n(-b_n) = \Delta_n(-b_3)$ And solution set $\Xi_n(b_n) = \Xi_n(b_3)$ The same, and their cardinalities are also equal: $\|\Xi_n(b_3)\| = \|\Delta_n(b_3)\| = \phi(G_3)\psi(L_3)$.

Therefore, we know from (1) (2) (3) (4) (5): $\Xi_n(b_3)$ and $\Xi_n(s(1))$ There is an inevitable connection between them, rather than alone and in isolation.

Figure 10: In $\Delta_3(b_3)$ At least one of $\delta_3(b_3)_{\min}: 1 \leq \delta_3(b_3)_{\min} \leq 2^{-i}(p_i^3 - p_j)$. Knowingly when $n = 3$ Time Theorem 2 holds.

3. The Present Inductive Hypothesis $n = k \geq 4$ is the Original Proposition "in $\Delta_k(b_k)$ At Least One of $\delta_k(b_k)_{\min}:

0 \leq |\delta_k(b_k)|_{\min} \leq 2^{-i}(p_i^3 - p_j) = 2^{-i}(\xi_{k-1}^2[2] - \xi_{k-1}[2]) = 2^{-i}(\xi_{k-1}^2[3] - \xi_{k-1}[3])$.

Established. When $n = k \geq 4$ period $p_n = p_{k-1} = \xi_{k-1}[2] = \xi_{k-3}[3] = \xi_{k-3}[4]$.

$\pi_n = \pi_k = p_1, p_2, \cdots, p_k = p_1, \xi_2[2], \xi_2[2], \xi_{k-1}[2] = p_1, \xi_2[2], \xi_{k-1}[2], \xi_{k-1}[2] = \xi_2[3], \xi_{k-1}[3], \xi_{k-1}[3], \xi_{k-3}[4]$.

$\pi'_n = p_2, p_3, \cdots, p_k = \xi_2[2], \xi_2[2], \xi_{k-1}[2] = \xi_2[2], \xi_{k-1}[2], \xi_{k-1}[2] = \xi_2[3], \xi_{k-1}[3], \xi_{k-1}[3], \xi_{k-3}[4]$, $b_k \in B_k = \{b_k | 1 \leq b_k \leq \pi_k'\}$.

$\Delta_n(b_n) = \Delta_k(b_k) = \{0 \leq \delta_n(b_n) < \pi'_n | \delta_k(b_k) \notin \pm b_k (\text{mod } p_i), i = 1, 2, \cdots, k\}$, for about $(b_k, \pi'_n) = G_k$, $1 \leq L_k = \pi'_n/G_k \leq \pi_k'$, $L_k$ you have $v_k$ Odd prime factors.

By 1.5 Module $\pi_n$ of $\pm B_s$ sieve $S_n[\pm B_s]$ knowing the laws:

3.1. Mole $\pi_n$ of $\pm B_s$ sieve $S_n[\pm B_s]$ Only the inside will have it

$2^{2-k} \prod_{i=2}^{k} (p_i + 1) = 2^{2-k} \prod_{i=2}^{k} (p_i + 1)$ class of sieve $S_k[\pm 0, 1, 2, \cdots, \phi(p_k)]$, $S_k[\pm 0, 1, 2, \cdots, \phi(p_k)]$.

$\pi_k$ of $\pm B_k$ which is the Solution set $\Delta_k(B_k)$ Inside and only $2^{2-k} \prod_{i=2}^{k} (p_i + 1)$

Knowing the module $\Delta_k(b_k) = \{0 \leq \delta_k(b_k) < \pi'_k | \pm b_k = \pm r_i (\text{mod } p_i), i = 1, 2, 3, \cdots, k\}$, if $\delta_k(b_k) = \pm r'_k \neq \pm r_i (\text{mod } p_i), (i = 1, 2, 3, \cdots, k)$ Knowing

$\delta_k(b_k) = \sum_{i=1}^{k} r_i(\pi_k/p_i)^{\phi(p_i)} = \delta_{k-1}(b_{k-1}) p_{k-1}^{\phi(p_i)} + r'_k \pi_k^{\phi(p_i)} (\text{mod } \pi_k)$, $\Delta_k(b_k)$ Cardinality

$\|\Delta_k(b_k)\| = \phi(G_k)\psi(L_k)$.

$L_k$ you have $v_k$ Odd prime factors, so: $S_k[\pm b_k]$ There are and only sieve $2^{n_k} \geq 1$ which is

3.2. Reasons the Kind of sieve, i.e. $S_k[\pm b_k]$ Set of sieve class $\Theta_k(b_k)$ The base is $\|\Theta_k(b_k)\| = 2^{n_k} \geq 1$. 

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3.3. Seven Knowing’s by Lemma, when

\[ b_k = \xi_k [\gamma] \in \Xi_k \text{ period } \Theta_k (\xi_k [\gamma]) = \xi_k [\gamma] \times \Theta_k (1). \]

3.4. Knowing from the theorem, if:

\[ \pi_n \text{ Special unary quadratic difference Group } \delta_n (b_n) \not\equiv \pm b_n \equiv \pm r_i \]

By 1.5 Module

\[ (\text{mod } p_i, \ i = 1,2,\cdots, n.) \]

The solution set law can be known, \( \Delta_n (b_n) \) and \( \Xi_n (b_n) \). Both cardinalities are equal:

\[ \| \Xi_n (b_n) \| = \phi(G_n) \psi(L_n). \]

By (1) (2) (3) (4) (5) knowing that: \( \Xi_n (b_n) \) and \( \Xi_n (1) \) There is an inevitable connection between them, rather than in isolation. As for \( n = k \) The inductive hypothesis of time: is known as \( \Delta_k (b_k) \) At least one of \( \delta_k (b_k)_{\min} \):

\[ 0 \leq | \delta_k (b_k) |_{\min} \leq 2^{-1} (p_k - p_k) = 2^{-1} (\xi_k^2 [2] - \xi_{k-1} [2]) = 2^{-1} (\xi_{k-2} [3] - \xi_{k-2} [3]). \]

So from the above formula, when serves as \( n = k \) \( \delta_k (b_k)_{\min} \) The range has already been all determine.

As when \( n = k + 1 \) period \( p_n = p_{k+1} = \xi_{(k+1)-1} [2] = \xi_{(k+1)-2} [3] = \xi_{(k+1)-3} [4], \)

\[ b_n = b_{k+1} \in B_{k+1} = \{ b_{k+1} \ | \ 1 \leq b_{k+1} \leq \pi'_{k+1} \} , \]

\[ \pi_n = \pi_{k+1} = p_n \delta_{n} \cdots p_2 \delta_2 \cdot \xi_{k+1-1} [2] = \xi_{(k+1)-2} [3] = \xi_{(k+1)-3} [4]. \]

\[ \pi_n' = \pi'_{k+1} = p_n \delta_{n} \cdots p_2 \delta_2 \cdot \xi_{k+1-1} [2] = \xi_{(k+1)-2} [3] = \xi_{(k+1)-3} [4]. \]

\[ \pm \Delta_n (b_n) = \pm \Delta_{k+1} (b_{k+1}) = \{ 0 \leq \delta_{k+1} (b_{k+1}) \leq \pi'_{k+1} | \delta_{k+1} (b_{k+1}) \not\equiv \pm b_{k+1} \equiv \pm r_i \ (\text{mod } p_i), \ i = 1,2,\cdots, k, k+1 \}, \]

Order: \( (b_{k+1}, \pi'_{k+1}) = G_{k+1}, 1 \leq L_{k+1} = \pi'_{k+1} / G_{k+1} \leq \pi'_{k+1}, L_{k+1} \) you have \( v_{k+1} \) as the odd prime factors.

By 1.5 mold. \( \pi_n \) of \( \pm B_n \) sieve \( \pm B_n \) the rule of law.

3.5. module \( \pi_{k+1} \) of \( \pm B_{k+1} \) is the sieve \( S_{k+1} [\pm B_n] \) that is inside and only

\[ 2^{n} \prod \limits_{i=2}^{n} (p_i + 1) = 2^{k+1} \prod \limits_{i=2}^{k+1} (p_i + 1) \ S_{k+1} [\pm \{0,1,0,2,\cdots, 0_k,0_{k+1}\}], \]

sieve:

\[ S_{k+1} [\pm \{0,1,0,2,\cdots, 0_k,0_{k+1}\}], \cdots S_{k+1} [\pm \{1,1,1,2,\cdots, \phi(p_{k+1}),2^{-1} \phi(p_{k+1})\}]. \]

3.6. \( \Delta_{k+1} (b_{k+1}) = \{ 0 \leq \delta_{k+1} (b_{k+1}) \leq \pi'_{k+1} | \delta_{k+1} (b_{k+1}) \not\equiv \pm b_{k+1} \equiv \pm r_i \ (\text{mod } p_i), \ i = 1,2,\cdots, k, k+1 \}. \]

If \( \delta_{k+1} (b_{k+1}) \not\equiv \pm r_i \ (\text{mod } p_i), \ i = 1,2,\cdots, k, k+1 \) knowing that

\[ \delta_{k+1} (b_{k+1}) = \sum \limits_{r=1}^{\pi_{k+1}} r^{(p_{k+1})} = \sum \limits_{r=1}^{\pi_{k+1}} r^{(p_{k+1})} + \sum \limits_{r=1}^{\phi(p_{k+1})} (\text{mod } \pi_{k+1}), \]

the solution set

Cardinality:

\[ \| \Theta_{k+1} (b_{k+1}) \| = \phi(G_{k+1}) \psi(L_{k+1}); \]

3.7. Follow by \( L_{k+1} \) you have \( v_{k+1} \) the Odd prime factors, is: \( S_{k+1} [\pm b_{k+1}] \) There are other factors but only sieve \( 2^{-n} \geq 1 \) the kind of sieve, which is: \( S_{k+1} [\pm b_{k+1}] \) Set of sieve class \( \Theta_{k+1} (b_{k+1}) \) The base is

\[ \| \Theta_{k+1} (b_{k+1}) \| = 2^{n} \geq 1. \]

3.8. The Seven Knowledge by Lemma, when

\[ b_{k+1} = \xi_{k+1} \in \Xi_{k+1} \text{ period is: } \Theta_{k+1} (\xi_{k+1}) = \xi_{k+1} \times \Theta_{k+1} (1). \]
3.9. Knowing from the theorem that, if \( n \geq 2 \), serves as \( \Xi_n(b_n) = \Xi_{k+1}(b_{k+1}) = \bigcup_{i=1}^{d(G_k+1)} \Xi_{k+1}(\xi_i'[\gamma_i]) \),
however:

By (1) (2) (3) (4) (5) we know: \( \Xi_{k+1}(b_{k+1}) \) and \( \Xi_{k+1}(1) \) There is an inevitable connection between them, rather than in isolation.

For \( n = k + 1 \), \( b_{k+1} \in B_{k+1} = \{ b_{k+1} | 1 \leq b_{k+1} \leq \pi'_k \} \) period: is \( \Delta_{k+1}(b_{k+1}) \) At least one of At least one of
\[
0 \leq |\delta_{k+1}(b_{k+1})| = 2^{-1}((\xi^2_{(k+1)-1}[3] - \xi_{(k+1)-1}[3]) - 2^{-1}(\xi^2_{(k+1)-1}[2] - \xi_{(k+1)-1}[2]) = 2^{-1}(p_{k+1}^2 - p_{k+1})
\]
when \( n = k + 1 \) period, Theorem two holds. From I, 2: When \( n \in N \) period, Theorem two holds.

[Theorem Three] When \( n \geq 1 \) fixed, module \( \pi_n \) is the even remainder \( 2b > p^2 \) Can be modeled as \( \pi_n \).

Two greater than \( p_n \) Unequal simplified sum is remaining. prove, when: \( n \geq 1 \)

fixed \( p_n \geq p_1 = 2 \), \( \pi_n = p_1p_2 \cdots p_n \), \( \Xi_n = \{ \xi_n | (\xi_n, \pi_n) = 1 \} \), \( \pi'_n = 2^{-1}\pi = p_2p_3 \cdots p_n \), \( B_n = \{ b_n | 1 \leq b_n \leq \pi'_n \} \), by the condition
\[
2^{-1}p_n^2 < b = t\pi_n + b_n \equiv b_n < \pi'_n \pmod{\pi'_n},
\]

From theorem two, for the module \( \pi'_n \) Any of the remaining classes \( b_n \). There must be at least one \( \delta_n(b_n)_{\text{min}} : 0 \leq |\delta_n(b_n)_{\text{min}}| \leq 2^{-1}(p_n^2 - p_n^2) < 2^{-1}p_n^2 < 2b \), knowing
\[
p_n < b \pm \delta_n(b)_{\text{min}} \equiv b_n \pm \delta_n(b)_{\text{min}} \in \Xi_n \pmod{\pi_n},
\]
for
\[
p_n < 2^{-1}(p_n^2 - p_n^2) - \delta_n(b)_{\text{min}} \leq b_n + t\pi_n \pm \delta_n(b)_{\text{min}} = b_n \pm \delta_n(b)_{\text{min}} < 2b,
\]
because
\[
p_1, p_2, \cdots, p_n \text{ all } \pi_n \text{ Factor instead of modulus } \pi_n \text{ is the simplified remainder of } 1 \text{ module } \pi_n \text{ is the Simplified remainder } \xi_n[k] \geq \xi_n[2] \geq \pi_{n+1} > p_n, \text{ Knowingly } n \geq 1 \text{ Theorem three holds.}

[Theorem 4] When \( n \geq 1 \), \( p_n^2 < 2b < p_{n+1}^2 \) fixed, even \( 2b \) Can be expressed as at least one pair (two) greater than \( p_n \) Odd prime number \( b \pm \delta_n(b)_{\text{min}} \) sum.

Prove: Three Theorems by Theorem: When \( n \geq 1 b_n \in B_n = \{ b_n | 0 \leq b_n \leq \pi'_n \} \) fixed, module \( \pi_n \) is the even remainder \( 2b_n \) to the table \( \pi_n \) A pair (two) greater than \( p_n \) Unequal simplification of the remaining sum. Knowing: when \( n \geq 1 \), \( p_n^2 \leq 2b < p_{n+1}^2 \) fixed \( b \in B_n \), Large even number \( 2b \) at least can be express as the module \( \pi_n \) Two of greater than 1 and less than \( p_{n+1}^2 \) Simplified remainder \( b \pm \delta_n(b)_{\text{min}} \) sum, reason \( p_n < b \pm \delta_n(b)_{\text{min}} < 2b < p_{n+1}^1 \), Therefore, we know from the prime discrimination method that these two \( p_n \) Unequal simplification of the remaining sum. Knowing: when \( n \geq 1 \), \( p_n^2 \leq 2b < p_{n+1}^2 \) fixed, \( b \in B_n \), Large even number \( 2b \) at least it can be express as the module \( \pi_n \) Two of greater than 1 and less than \( p_{n+1}^2 \) Simplified remainder \( b \pm \delta_n(b)_{\text{min}} \) sum, reasons \( p_n < b \pm \delta_n(b)_{\text{min}} < 2b < p_{n+1}^1 \). Therefore, we know from the prime discrimination method that these two module \( \pi_n \) is the Simplified remainder \( b \pm \delta_n(b)_{\text{min}} \) which is for \( 2b_n \) Greater than \( p_n \) A pair of odd prime numbers. Therefore, the original proposition is true, that is, the Geerbach conjecture is true.

Example 7: sum \( 2b = 72 \) Greater than \( \sqrt{2b} = \sqrt{72} > 7 \) The smallest two primes apart.

Solution method 1: \( \therefore p_4^2 = 49 < 2b_4 = 72 < p_5^2 \). obtain module

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\[\pi_n = \pi_4 = p_1 p_2 p_3 p_4 = 210.\] And know from the conditions: \(b = 4 = 36\) as for

\[G_4 = (b, \pi_4) = (36, 3 \times 5 \times 7) = 3 L_4 = \pi_4 | G_4 = 35\]

\[C_4 = \{c | (c, G_4) = 1\} = \{c | (c, 3) = 1\} = \{c | [c] = 1, \pi_4[2] = 2\}.\] Knowing by Lemma:

\[b_i \pm c_i L_4 \equiv \xi_i \in \Xi_4, \text{ about module } \pi_4 = 210 \xi_i[\alpha_i] = 36 + 1 \times 35 = 71\]

\[\xi_i[\alpha_2] = 36 + 2 \times 35 = 106 \equiv -104 \pmod{\pi_4}\]

From Table 1 (see attached table), you can find:

\[\Delta_4(71) = \{0,12,18,30,42,60,72,102\}, \Delta_4(106) = \{3,33,45,63,75,87,93,105\}\]

Odd: \(\Delta_4(36) \subseteq \Delta_4(71) \cup \Delta_4(106) \quad \Delta_4(36) = \{1,11,13,19,29,31,41,43,53,59,61,71,73,83,89\}\]

one of them the reasons:

\[36 - \xi_i(\alpha) \in P(36), 2 \times 36 - \xi_i(\alpha) \in P(36).\] So at least one \(\xi_i(\alpha_i) = 29 \in \Omega_72, 29 = 43 \in P(36)\)

Reasons: \(36 - \xi_i(\alpha_i)\)

So at least one \(\xi_i(\alpha_i) = 31 \in \Omega_472, 31 = 41 \in P(36)\). Know by comparison. 72 The smallest pair of primes apart is 31 and 41.

Solution method 2: reasons, \(p^2 = 49 \times 2b = 72 \times p_5 = 121\) Selection model \(\pi_n = \pi_4 = 210\)

\(b_4 = 36 \equiv \{0,1,2,1,1,4\} \pmod{\pi_4}\) for \(2^{-1}(p^2 - 1) = 2^{-1}(7^2 - 1) = 23\)

Which is half open interval \((0, 24)\) Middle crop sieve diagram (Figure 11)

\[S_4[\pm b_4] = S_4[\pm 36] = S_4[\pm 0,1,2,1,1,4] \rightarrow \]

Figure 11. \(S_4[\pm b_4] = S_4[\pm 36] = S_4[\pm 0,1,2,1,1,4] \rightarrow\)

According figure 11: \(\Delta_4(b_n) = \Delta_4(36) \Rightarrow \{5,7,17,23\}\) which is at least one of

\[0 \leq \delta_4(36) \leq 5 \leq 2^{-1}(p_5^2 - 1) \leq 24\] Know that at least there is for \(2b = 72\)

The pair of (smallest distances) primes are: \(b_4 - \delta_4(b_4) = 36 \pm 5 = \{41, 31\}\)

\(n = 4\) fixed, set \(\Xi_4(b_n) = \Xi_4(\pi_4)\) is the series. (Symbol description in the table:

Enclose: when

\(b_i \in B_4 = \{b_i | 1 \leq b_i \leq \pi_4\}, G_4 = (b, \pi_4), b_i = m_i G_4, L_4 = \pi_4 | G_4\), \n
\[Z_4 = \{\xi_i | \xi_i(b_4 - 1), \pi_4 = 1, (\xi_i, 2) = 1\}, W_4 = \{w_4 | (w_4(w_4 - 1), L_4) = 1, (w_4, 2G_4) = 1\} \subseteq Z_4,\]

\[\Xi_4(b_i) = \{\xi_i(b_i) | \xi_i(b_i) \equiv (2b_i - L_4) w_4 \pmod{\pi_4}\}\]

3.10. when \(G_4 = (b, \pi_4) = 1, L_4 = \pi_4 = 105\) fixed \(b_i \in \Xi_4 = \{1,2,4,8,11,13, \cdots, 103,104\}\). 

\(\xi_i(b_i) = \xi_i(\xi_i) = (2b_i - \pi_4) \xi_i \pmod{\pi_4},\]
\[ \zeta_4 \in Z_4 = \{ \zeta_4 \mid (\zeta_4, \zeta_4 - 1, \pi_4) = 1 \} = \{13, 17, 19, 23, \ldots, 193, 199, 209\} \]

**Enclose 1:** \( \Xi_4(b_4) = \Xi_4(\zeta_4) \)

![Figure 12](image12.jpg)

**Figure 12.** \( \Xi_4(b_4) = \Xi_4(\zeta_4) \)

### 3.11 When \( G_4 = (b_4, \pi_4) = 3, L_4 = 5 \times 7 = 35 \) fixed:

- \( m_4 \in M_4 = \{ m_4 \mid (m_4, L_4) = 1 \} = \{1, 2, 3, 4, 6, 8, \ldots, 32, 33, 34\} \).
- \( b_4 = m_4 G_4 = 3m_4 \in \{3, 6, 9, 12, 18, 24, \ldots, 99, 102\} \).
- \( w_4 \in W_4 = \{ w_4 \mid (w_4, L_4 - 1) = 1 \} \).
- \( \xi_4(b_4) \equiv (2b_4 - L_4) w_4 \pmod{\pi_4} \).

**Enclose 2:** \( \Xi_4(b_4) \)

![Figure 13](image13.jpg)

**Figure 13.** \( \Xi_4(b_4) \)

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3.12. When $G_4 = (b_4, \pi'_4) = 5$ time:
$L_4 = 3 \times 7 = 21$,
$m_4 \in M_4 = \{m_4 \mid (m_4,3 \times 7) = 1\} = \{1,2,4,5,8,10,11,13,16,17,19,20\}$,
\[b_4 = m_4 G_4 = 5 m_4 = \{5 \times 10, 20, 25, 40, 50, 55, 65, 80, 85, 90, 100\},\]
\[W_4 = \left\{ w_4 \mid (w_4 - 1, L_4) = 1, \quad (w_4, \quad 2G_4) = 1. \right\}\]
\[= \{1,17,23,41,47,\ldots,179,191,209\}, \Xi_4(b_4) = \left\{ \xi_4(b_4) \mid \xi_4(b_4) \equiv (2b_4 - L_4) w_4 \pmod{\pi_4} \right\} \]

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![Figure 14. $\Xi_4(b_4)$](image1)

3.13. When, $G_4 = 7, L_4 = 3 \times 5 = 15$
$m_4 \in M_4 = \{m_4 \mid (m_4,3 \times 5) = 1\} = \{1,2,4,7,8,11,13,14,\ldots\}$,
\[b_4 = m_4 G_4 = 7 m_4 = \{7,14,28,49,56,77,91,98,\ldots\},\]
\[= \{11, 17, 23, 41, 47, 53, 83, 89, 101, 107, \ldots\ \}
\[\Xi_4(b_4) = \left\{ \xi_4(b_4) \mid \xi_4(b_4) \equiv (2b_4 - L_4) w_4 \pmod{\pi_4} \right\} \]

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![Figure 15. $\Xi_4(b_4)$](image2)

3.14. When $G_4 = 5 \times 7 = 35, L_4 = 3 \times 5 = 15$
$m_4 \in M_4 = \{m_4 \mid (m_4,3 \times 5) = 1\} = \{1,2,3,4,5,6,\ldots\}$,
\[b_4 = m_4 G_4 = 35 m_4 = \{35,70,\ldots\},\]
\[W_4 = \left\{ w_4 \mid (w_4 - 1, L_4) = 1, \quad (w_4, \quad 2G_4) = 1. \right\}\]
\[= \{11,17,23,29,41,47,53,59,71,83,89,101,107,113,\ldots\ \}
\[\Xi_4(b_4) = \left\{ \xi_4(b_4) \mid \xi_4(b_4) \equiv (2b_4 - L_4) w_4 \pmod{\pi_4} \right\} \]

(The omitted schedule.)

5. When $G_4 = 3 \times 5 = 15, L_4 = 7$,
$m_4 \in M_4 = \{1,2,3,4,5,6,\ldots\}$,
\[b_4 = m_4 G_4 = 15 m_4 = \{15,30,45,60,75,90,\ldots\}, \quad W_4 = \left\{ w_4 \mid (w_4 - 1, L_4) = 1, \quad (w_4, \quad 2G_4) = 1. \right\}\]
\[
\{1, 13, 17, 19, 23, 31, 37, 41, 47, 53, 59, 61, 67, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 121, 131, 137, 139, 143, 149, 151, 157, 163, 167, 173, 179, 181, 187, 191, 193, 199, 209.\}
\]
\[
\xi_4(b_4) \equiv (2b_4 - L_4)\nu_4 \pmod{\pi_4}. \quad \text{(The omitted schedule)}
\]

References

