The Nonlocal Problem of a Fractional Nonlinear Differential Equation with Nonlinear Growth Condition

Shanshan Gao\textsuperscript{1, 2, b} and Yi Cheng\textsuperscript{1, a, *}

\textsuperscript{1} Department of Mathematics, Bohai University, Jinzhou, Liaoning, China,
\textsuperscript{2} Department of information technology, Liaoning University of Science and Engineering, Jinzhou, Liaoning, China.

\textsuperscript{a} E-mail: 645924181@qq.com.
\textsuperscript{b} E-mail: gaoshanshan0218@126.com.
\textsuperscript{*} corresponding author

Keywords: fractional differential equation; nonlocal condition; nonlinear growth condition; Leray-Schauder fixed theorem

Abstract: This paper considers the nonlocal problem of a fractional nonlinear differential equation with nonlinear growth condition. We use the Leray-Schauder fixed theorem and some fractional inequalities, to prove the existence and uniqueness of solution for the fractional order differential system. Finally, we prove that the fractional nonlinear differential equation with nonlinear growth condition has a unique solution.

1. Introduction

The non-local boundary value problem of differential equation means that the definite solution condition of the differential equation is related to both the points in the solution interval and the whole solution interval. In recent years, differential equations or differential inclusions under non-local conditions have attracted more and more attention. This kind of non-local condition contains many boundary value conditions, such as initial value, period, counter-period, rotation period, integral, multi-point average, etc. Therefore, this condition is more general and more extensive in practical application.

It is found that fractional calculus can use fewer parameters to describe some complex phenomena. At present, fractional calculus theory is widely used in various fields of natural science, such as signal processing, image processing, anomalous diffusion, random process, physical mechanics, biomedicine and automatic control. Moreover, the nonlocality of fractional differential equation is more suitable for describing memory and heredity, which makes fractional differential equation more widely studied and applied.

Due to its wider application, the nonlocal Cauchy problem has attracted more and more scholars’ interest. For instance, in\textsuperscript{[1]}, Boucherif al. proved that the semilinear evolution equations with nonlocal initial conditions at least exist one solution. The existence of co-solutions for a class of nonlinear evolution equations subjected to nonlocal initial conditions is studied by Angela Paicu al. in \textsuperscript{[2]}. In Banach spaces, Sergiu Aizicovici al.\textsuperscript{[3]} considered some multivalued equations which under nonlocal initial conditions. However, it is worth noting that these works mentioned above considered the integral order differential systems, few results on the fractional differential systems under the nonlocal conditions. In \textsuperscript{[4]}, using Banach contraction mapping principle, Bashir Ahmad al. proved that a multi-term fractional differential equation with generalized integral boundary conditions has a unique solution.

In our text, a fractional nonlinear differential equation with nonlinear growth condition as following is studied by us:
\[
\begin{cases}
^cD^\alpha z(t) + Hz(t) = G(t, z(t)) + f(t), & \text{for a.a. } t \in I := [0, T] \\
z(0) = \varphi(z)
\end{cases}
\] (1.1)

where \( \alpha \in (0,1) \) and \( H : R^N \rightarrow R^N \) is a linear operator, \( G : I \times R^N \rightarrow R^N \) is a function, which is nonlinear. \( f \in L^p(I, R^N) \) is a measurable function, \( \varphi : C(I, R^N) \rightarrow R^N \) is the nonlocal function. Here \( ^cD^\alpha \) denotes the Caputo fractional derivative with respect to t, which is defined as:

when \( \alpha = n \),

\[
^cD^\alpha z(t) = z^{(n)}(t);
\]

when \( \alpha \in (n-1, n) \),

\[
^cD^\alpha z(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\sigma)^{n-\alpha-1} z^{(n)}(\sigma)d\sigma,
\] (1.2)

where \( n \in N \) and \( \Gamma(\cdot) \) is the Gamma function. We use the Leray-Schauder alternative theorem to attest the existence theory of the solution to equation (1.1), and then apply some important inequalities of fractional calculus to get the uniqueness of the solution. For more knowledge and properties on the fractional calculus, we refer the readers to see [5,6,7].

This paper is arranged as following. Some lemmas for the fractional calculus and the main result are given in Section 2, and the certification of the theorem is presented in Section 3.

2. Preliminaries and Main result

Recalling some lemmas which act a pivotal part in our work.

2.1 Lemma [8]

Let \( Q \) be a subset of \( X \), which \( X \) be a Banach space. \( Q \) is nonempty, bounded and convex. \( \Omega \) is an open subset of \( Q \) with \( 0 \in \Omega \). Let \( \Psi : \overline{\Omega} \rightarrow Q \) is a continuous, compact map. Then either there exist \( y \in \partial \Omega \) and \( \beta \in (0,1) \) with \( y = \beta \Psi(y) \) or \( \Psi \) has a fixed point \( y \in \Omega \).

2.2 Lemma [9]

Let us assume that \( \chi(\tau) \in R^N \) is a function, which is continuous and differentiable, and \( P \in R^{N \times N} \) is a positive definite matrix. Then

\[
\frac{1}{2}^cD^\alpha [\chi^T(\tau)P\chi(\tau)] \leq \chi^T(\tau)P^\alpha D^\alpha \chi(\tau) \quad \forall \alpha \in (0,1).
\]

2.3 Lemma [10]

Let us suppose that \( S : R \rightarrow R^+ \) is a function, which is continuous, and the function such that

\[
^cD^\alpha S(t) \leq - \mu S(t) \quad \forall \alpha \in (0,1),
\]

where \( \mu > 0 \) is a constant. Then

\[
S(t) \leq S(0)E_\alpha (-\mu t^\alpha) \quad \forall t \geq 0,
\]

where \( E_\alpha(u) := \sum_{k=0}^{\infty} \frac{u^k}{\Gamma(\alpha k + 1)} \), \( \forall u \in R \), is the Mittag-Leffler function. Throughout this paper, the norm of \( C(I, R^N) \) is denoted by \( \|z\| = \max_{t \in I} |z(t)| \), the norm of Banach space \( L^p(I, R^N) \) is expressed by \( \|z\| = \left( \int_0^T |z(t)|^p dt \right)^{1/p} \) with \( p > \frac{1}{\alpha} \), and the inner product in \( R^N \) is represented by \( \langle \cdot, \cdot \rangle \).
Let \( \Omega_\rho := \{ \mathbf{y} \in C([0,T]; R^N) : \| \mathbf{y} \| < \rho \} \), where \( \rho \) is a positive constant. It is easy to know \( \Omega_\rho \) is open subset of \( C([0,T]; R^N) \), and \( \Omega_\rho \) is bounded.

In order to put forward our main result, the following hypotheses are supposed to hold all over the paper:

(H1): \( H : R^N \to R^N \) is a bounded linear operator satisfying \( \langle Hz, \mathbf{z} \rangle \geq c \| \mathbf{z} \|^2 \), \( \forall \mathbf{z} \in R^N \), where \( c > 0 \) is a constant.

(H2): \( G : I \times R^N \to R^N \) is a nonlinear function that makes

(i) for every \( t \in I, u \in R^N, t \to G(t,u) \) is measurable and \( u \to G(t,u) \) is continuous;

(ii) there is a measurable function \( g(t) \in L^p(I) \) and a nondecreasing continuous function \( \psi : R^+ \to R^+ \) that makes \( \|G(t,\mathbf{z})\| \leq g(t)%\psi(\|\mathbf{z}\|) \), for \( t \in I \), every \( \mathbf{z} \in C(I; R^N) \);

(iii) there is a function \( v \in L^p(I, R^+) \) that makes

\[ \langle G(t,\mathbf{z}_1) - G(t,\mathbf{z}_2), \mathbf{z}_1 - \mathbf{z}_2 \rangle \leq v(t) \| \mathbf{z}_1 - \mathbf{z}_2 \|^2, \quad \forall \mathbf{z}_1, \mathbf{z}_2 \in R^N, \]

for a.a. \( t \in I \), where \( \|\mathbf{z}\| \leq c \) with the constant \( c > 0 \) given in (H1).

Thanks to [11], we know that the solution of (1.1) can be expressed as following:

\[
\begin{align*}
\mathbf{z}(t) &= E_a(\mathbf{H}t^\alpha)\mathbf{z}(0) + \int_0^t (t-s)^{\alpha-1} E_{a,a}(\mathbf{H}(t-s)^\alpha) G(s, \mathbf{z}) + f(s) ds, \\
\end{align*}
\]

where \( E_{a,\beta}(z) := \sum_{k=0}^\infty \frac{z^k}{\Gamma(a k + \beta)} \), \( \forall z \in R \), is the generalized Mittag-Leffler function in two parameters. It is well known that the Mittag-Leffler function is continuous, we let \( M_E = \max_{a} \| E_a(\mathbf{H}t^\alpha) \|, \bar{M}_E = \max_{a} \| E_{a,a}(\mathbf{H}t^\alpha) \| \) for convenience.

### 2.4 Lemma

If \( f \in L^p([0,T], R^N) \) and the hypothesis (H2) hold, the following estimates are valid:

\[
\begin{align*}
\int_0^t (t-\sigma)^{\alpha-1} | f(\sigma) | d\sigma &\leq M_1, \quad t > 0, \\
\int_0^T (t-\sigma)^{\alpha-1} \| G(\sigma, \mathbf{z}(\sigma)) \| d\sigma &\leq M_2, \quad t > 0,
\end{align*}
\]

where \( M_1, M_2 > 0 \) are constants independent of \( t \).

**Proof:** Applying the Hölder’s inequality, it is easy to get:

\[
\begin{align*}
\int_0^T (t-\sigma)^{\alpha-1} | f(\sigma) | d\sigma &= \left[ \int_0^T (t-\sigma)^{\alpha(\alpha-1)} d\sigma \right]^\frac{1}{\alpha} \left[ \int_0^T | f(\sigma) |^\rho d\sigma \right]^\frac{\rho}{\alpha} \\
&\leq \left[ \frac{\Gamma(\alpha(\alpha-1)+1)}{q(\alpha-1)+1} \right]^\frac{1}{\alpha} \left[ \int_0^T | f(\sigma) |^\rho d\sigma \right]^\frac{\rho}{\alpha} \\
&\leq \left[ \frac{T^{\alpha(\alpha-1)+1}}{q(\alpha-1)+1} \right]^\frac{1}{\alpha} \left[ \int_0^T | f(\sigma) |^\rho d\sigma \right]^\frac{\rho}{\alpha} \\
&= M_1,
\end{align*}
\]

where \( q = \frac{p}{p-1} \). In light of hypothesis (H2)(ii), and in the similar way, for any \( \mathbf{z} \in \Omega_\rho \), there is a constant \( M_2 > 0 \) that makes
\[ \int_{0}^{t} (t - \sigma)^{\alpha-1} \|G(\sigma, z(\sigma))\| d\sigma \leq \int_{0}^{t} (t - \sigma)^{\alpha-1} g(\sigma) \psi(\|z\|) d\sigma \leq \left[ \int_{0}^{t} (t - \sigma)^{\alpha-1} d\sigma \right]^\frac{1}{\gamma} \left[ \int_{0}^{t} \|g(\sigma)\|^{\gamma} d\sigma \right]^\frac{1}{\gamma} \psi(\|z\|) \leq \frac{T^{\gamma(\alpha-1)} + 1}{q(\alpha-1) + 1} \left[ \int_{0}^{T_t} \|g(\sigma)\|^{\gamma} d\sigma \right]^\frac{1}{\gamma} \psi(\rho) = M_2. \] (2.3)

The proof is complete.

In order to get our main result, the following assumptions are also essential:

(H3): \( \phi : C([t, T] ; R^N) \rightarrow R^N \) is a continuous function such that

(i) there exists a nondecreasing function \( h : R^+ \rightarrow R^+ \), satisfying \( |\phi(z)| \leq h(\rho) \), for any \( z \in \Omega_\rho \);

(ii) There exists a constant \( 0 < l < 1 \), for \( t \in [0, T] \) such that

\[ \phi : |\phi(x) - \phi(y)| \leq l \|x - y\|, \quad \text{for every } x, y \in R^N. \]

(H4): There exists a constant \( \rho > 0 \) such that

\[ M_1 h(\rho) + \hat{M}_1 M_2 \psi(\rho) + \hat{M}_1 M_1 < \rho, \]

where \( M_1, \hat{M}_1, M_1, M_2, M_1 \) are constants given above. Now, we present the main result:

**Theorem 1** If the hypotheses (H1), (H2), (H3), (H4) are satisfied, the equation (1.1) has a unique solution \( z \in C([t, T] ; R^N) \).

### 3. Proof of Theorem 1

To testify the existence of solution to the equation (1.1), we introduce the operator

\[ T : C([t, T] ; R^N) \rightarrow C([t, T] ; R^N) \]

which is defined as follows:

\[ T(z) = E_\alpha \left( Ht^\alpha \right) \phi(z) + \int_{0}^{t} (t - \sigma)^{\alpha-1} E_{\alpha, \sigma} \left( H(t - \sigma)^\alpha \right) G(\sigma, z) + f(\sigma) d\sigma, \] (3.1)

and then, we establish in several steps to certificate that the operator \( T \) has a fixed point.

#### 3.1 Step 1. \( T \) is continuous.

Set \( \{ z_{\omega} \} \) be a sequence such that \( z_{\omega} \rightarrow z \) in \( C([0, T] ; R^N) \), as \( \omega \rightarrow \infty \). Then,

\[ \|T(z_{\omega}) - T(z)\| \leq \left\| E_\alpha \left( Ht^\alpha \right) \phi(z_{\omega}) - \phi(z) \right\| + \left\| \int_{0}^{t} (t - \sigma)^{\alpha-1} E_{\alpha, \sigma} \left( H(t - \sigma)^\alpha \right) G(\sigma, z_{\omega}) - G(\sigma, z) d\sigma \right\|. \] (3.2)

In light of hypothesis (H2)(i) and (H3)(i), it follows that

\[ \|G(\sigma, z_{\omega}) - G(\sigma, z)\| \rightarrow 0, \|\phi(z_{\omega}) - \phi(z)\| \rightarrow 0, \quad \text{as } z_{\omega} \rightarrow z, \]

which implies \( \|T(z_{\omega}) - T(z)\| \rightarrow 0, \quad \text{as } z_{\omega} \rightarrow z, \) that is, \( T \) is continuous.

#### 3.2 Step 2. \( T \) is equicontinuous.

Let \( z \in \Omega_\rho \), for any \( t_u, t_v \in [0, T] \) with \( t_u < t_v \), we can infer that
Because of (2.2), (2.3), and the continuity of Mittag-Leffler function, we can derive that
\[ T(z(t_1)) - T(z(t_0)) \to 0 \quad \text{as} \quad t_0 \to t_1, \quad \forall z \in \overline{\Omega}_\rho. \] This means \( T \) is equicontinuous.

3.3 Step3. \( T \) maps bounded sets into bounded sets.

For each \( z \in \overline{\Omega}_\rho, t \in [0, T] \), owing to (H2)(ii), (H3)(i) and H(4), one can gain
\[
\| T(z) \| = \left\| E_{\alpha\beta} \left( H_{\alpha\beta} \right) \rho + \int_0^t (t-s)^{\alpha-1} E_{\alpha\beta} \left( H(t-s)^{\alpha} \right) G(z, s) + f(s) \, ds \right\|
\leq \left\| E_{\alpha\beta} \left( H_{\alpha\beta} \right) \rho \right\| + \left\| \int_0^t (t-s)^{\alpha-1} E_{\alpha\beta} \left( H(t-s)^{\alpha} \right) G(z, s) + f(s) \, ds \right\|
\leq M_E h(\rho) + \tilde{M}_E M_2 \psi(\rho) + \tilde{M}_E M_1
\leq \rho,
\]
which is the desired result.

3.4 Step4. \( T \) has a fixed point.

Suppose \( z \in \partial \Omega_\rho \), there is \( \beta \in (0,1) \) that makes \( z = \beta T(z) \). It then follows from (3.4) that
\[
\| T(z) \| = \beta \| T(z) \|
\leq \beta \left( M_E h(\rho) + \tilde{M}_E M_2 \psi(\rho) + \tilde{M}_E M_1 \right)
\leq M_E h(\rho) + \tilde{M}_E M_2 \psi(\rho) + \tilde{M}_E M_1
< \rho.
\]

Obviously, it is a contradiction. By applying Lemma 2.1, the operator \( T \) has a fixed point. Now, it suffices to say that the equation (1.1) has at least one solution. Next, we will prove the uniqueness of solution to equation (1.1). Assume \( z_u, z_v \in C([0, T], \mathbb{R}^N) \) are the solutions to equation (1.1), and bring them into (1.1), we can get
\[
\langle D^\alpha z_u(t) + H z_u(t) \rangle = G(t, z_u(t)) + f(t),
\]
\[
\langle D^\alpha z_v(t) + H z_v(t) \rangle = G(t, z_v(t)) + f(t).
\]
Subtracting the above two formulas, and taking an inner product above with \( z_u - z_v \), we deduce that
\[
\langle D^\alpha (z_u - z_v), z_u - z_v \rangle + \langle H (z_u - z_v), z_u - z_v \rangle = \langle G(t, z_u) - G(t, z_v), z_u - z_v \rangle.
\]
In light of Lemma 2.2, (H1) and (H2)(iii), one can get
\[
\frac{1}{2} D^\alpha \|z_u - z_v\|^2 + c \|z_u - z_v\|^2 \leq v(t) \|z_u - z_v\|^2,
\]
i.e.
\[
\zeta D^\alpha \|z_u - z_v\|^2 \leq 2(v(t)-c) \|z_u - z_v\|^2. \tag{3.6}
\]
Invoking Lemma 2.3, it follows that
\[
\|z_u - z_v\|^2 \leq \|\phi(z_u) - z_v(0)\|^2 E_\alpha \left(2\left(\|v(t)\|_\infty - c\right)^\alpha \right), \tag{3.7}
\]
which together with \( z_v(0) = \phi(z_u), z_v(0) = \phi(z_v) \), and (H3)(ii) deduces that
\[
\|z_u - z_v\|^2 \leq \|\phi(z_u) - \phi(z_v)\|^2 E_\alpha \left(2\left(\|v(t)\|_\infty - c\right)^\alpha \right)
\leq l^2 \|z_u - z_v\|^2 E_\alpha \left(2\left(\|v(t)\|_\infty - c\right)^\alpha \right)
\]
i.e. 
\[
\left[ 1 - l^2 E_\alpha \left(2\left(\|v(t)\|_\infty - c\right)^\alpha \right) \right] \|z_u - z_v\|^2 \leq 0.
\]
Owing to hypothesis (H2)(iii) and the monotonousness of the Mittag-Leffler function, we infer 
\[
E_\alpha \left(2\left(\|v(t)\|_\infty - c\right)^\alpha \right) < 1.
\]
Since (H2)(iii), we can infer that \( \|z_u - z_v\|^2 = 0 \), which implies \( z_u = z_v \). This allows to say that equation (1.1) has a solution and the solution is unique. This proof is thus complete.

References


[10]. J. Chen, Z. Zeng, and P. Jiang, “Global mittag-leffler stability and synchronization of